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Quadratic degenerations of odd-orthogonal Schubert varieties

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ABSTRACT

This paper is the second in a series leading to a type B_n geometric Littlewood–Richardson rule. The rule will give an interpretation of the B_n Littlewood–Richardson numbers as an intersection of two odd-orthogonal Schubert varieties and will consider a sequence of linear and quadratic deformations of the intersection into a union of odd-orthogonal Schubert varieties. This paper describes the setup for the rule and specifically addresses results for quadratic deformations, including a proof that at each quadratic degeneration, the results occur with multiplicity one. This work is strongly influenced by Vakil's [14].

This paper is the second in a series that will give a geometric rule for the odd-orthogonal Littlewood–Richardson numbers. The first, [5], proves a B_n version of Vakil's [14] Lemma 5.4. This result, restated here in Lemma 2, about the codimension of particular subvarieties of a Schubert variety described by containment, plays a critical role in the proofs of the current paper. In this paper, we lay the groundwork for the heart of the geometric rule for odd-orthogonal Littlewood–Richardson numbers. A specialization order in the double orthogonal flag variety involving codimension one linear and quadratic degenerations is given, the initial setup for the intersection of two Schubert varieties is described, and explicit results for the quadratic degenerations are proved, including proof of multiplicity one.

An outline of the paper is as follows:

1. Section 1 gives a brief review of the B_n setting and sets notation for later use.
2. Section 2 begins with a statement of the B_n Littlewood–Richardson Rule (Conjecture 1), presents a specialization order in the double orthogonal flag variety (Section 2.1), and describes the initial setup for the intersection of two isotropic Schubert varieties (Section 2.2).
3. Section 3 states the combinatorial rule for quadratic degenerations.
4. Section 4 gives a geometric proof for quadratic degenerations. Section 4.3 states definitions and results that hold for both linear and quadratic degenerations. In particular, we show that the base space T is reduced and irreducible. Section 4.4 focuses on the results that apply directly to quadratic degenerations and completes the proof of the combinatorial rule for quadratic degenerations.

1. B_n background

The type B_n Littlewood–Richardson numbers are the structure coefficients for the ring of symmetric functions with basis of Hall-Polynomials or P-polynomials [2,12,13,15]. Geometrically, the type B_n Littlewood–Richardson numbers are structure coefficients of the cohomology ring of the odd-orthogonal (type B_n) Grassmannian [1,8,10].

We give a short review of the geometric B_n-setting. For a more thorough treatment, see [7]. The (odd) orthogonal Grassmannian, $OGr(k, 2n+1)$, is created by placing a non-degenerate symmetric bilinear form B on the odd-dimensional

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vector space $\mathbb{C}^{2n+1}$. Without loss of generality, we let

$$B(x,y) = \begin{bmatrix} x_1 & \dots & x_{2n+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{2n+1} \end{bmatrix}$$

for $x, y \in \mathbb{C}^{2n+1}$.

A subspace $V \subset \mathbb{C}^{2n+1}$ is called *isotropic* if $B(x,y) = 0$ for all $x, y \in V$ and is *maximal isotropic* if further, $\dim(V) = n$. For a subspace, V, its *perp space* or *perp* is

$$V^{\perp} = \left\{ x \in \mathbb{C}^{2n+1} \mid B(x,y) = 0 \; \forall y \in V \right\}.$$

An *isotropic flag*, $F_.$, is a flag $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{2n+1} = \mathbb{C}^{2n+1}$, where for $1 \leq i \leq n$, F_i is isotropic and $F_{2n+1-i} = F_i^{\perp}$. The variety of all such isotropic flags is the *(odd) orthogonal flag variety*, $OFl(2n+1)$. The variety consisting of all k-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$ is $OGr(k, 2n+1)$, the *orthogonal Grassmannian*. The *maximal* orthogonal Grassmannian, $OGr(n, 2n+1)$, has a cell structure (with respect to an isotropic flag $F_.$) consisting of 2^n cells. We call these cells type B_n *Schubert cells*, and their closures *Schubert varieties*. Such a cell is denoted $\Omega_\lambda(F_.)$ where λ is a strict partition with length $\ell(\lambda)$ and no part larger than n. Let $j = (j_1 < j_2 < \cdots < j_n)$ be the set of jumping numbers for $\Omega_\lambda(F_.)$. That is, $\dim(V \cap F_{j_i}) = \dim(V \cap F_{j_i - 1}) + 1$ for $1 \leq i \leq n$. Then

$$\Omega_\lambda(F_.) = \{V \in OGr(n, 2n+1) \mid j_i = n + 1 - \lambda_i \text{ for } 1 \leq i \leq \ell(\lambda) \text{ and } j_i > n+1 \text{ if } i > \ell(\lambda)\}.$$

The Schubert variety $\overline{\Omega_\lambda}(F_.)$ is then

$$\overline{\Omega_\lambda}(F_.) = \{V \in OGr(n, 2n+1) \mid \dim(V \cap F_{j_i}) \geq i \; \forall i\}.$$

For a fixed flag $F_. \in OFl(2n+1)$, the Schubert cells give rise to the set of Schubert classes,

$$\{\tau_\lambda \in H^*(OGr(n, 2n+1), \mathbb{Z}) \mid \lambda \in \mathcal{P}_n\}$$

which form an additive basis for the cohomology of the orthogonal Grassmannian, or equivalently, for its Chow ring[1] [7,11]. So multiplying two Schubert cycles via the cup product yields a linear combination of Schubert cycles.

$$\tau_\lambda \tau_\mu = \sum_{\nu \in \mathcal{P}_n} a^{\nu}_{\lambda\mu} \tau_\nu$$

for some integers $a^{\nu}_{\lambda\mu}$. These integers are the *type B_n Littlewood–Richardson numbers*.

The goal of this paper is to present a geometric interpretation of the type B_n Littlewood–Richardson numbers. This will be done using degenerations. Degeneration methods have long been employed to answer enumerative problems [9]. For a more detailed summary of recent work that uses degenerations to describe structure constants of the cohomology of homogeneous spaces, see [4]. In [14], Vakil presents a geometric rule for the A_n Littlewood–Richardson numbers. The ideas in this paper are strongly influenced by his work.

2. The setup

The geometric interpretation presented in this paper takes two transverse isotropic flags $(M_., F_.) \in OFl(2n+1) \times OFl(2n+1)$ and describes a path through $OFl(2n+1) \times OFl(2n+1)$ to the diagonal by deforming $M_.$ until it coincides with $F_.$. This is described by the *specialization order* and is represented with black checkers on a square checker board.

Given two Schubert varieties, we follow the deformations of the intersection $\overline{\Omega_\lambda}(F_.) \cap \overline{\Omega_\mu}(M_.)$ across the specialization order path. For a given $(F_., M_.)$ in the specialization order, $[V]$ is represented by white checkers on the same checker board. The goal is to understand how $[V]$ deforms and to accurately represent this combinatorially.

Conjecture 1. *Let τ_λ and τ_μ be Schubert classes for the orthogonal Grassmannian $OGr(n, 2n+1)$, with $\tau_\lambda \tau_\mu = \sum a^{\nu}_{\lambda\mu} \tau_\nu$. Then the coefficient $a^{\nu}_{\lambda\mu}$ is equal to the number of isotropic checker games with input λ, μ and output ν.*

2.1. Specialization order

Consider a $(2n+1) \times (2n+1)$ checker board with $2n+1$ black checkers, no two in the same row or column. We make a rank table describing $\dim(M_i \cap F_j)$ such that $\dim(M_i \cap F_j)$ is equal to the number of black checkers weakly northwest of position (i, j).

Definition 1.

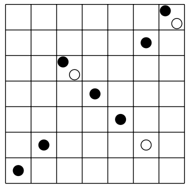
$$X_\bullet = \{(M_., F_.) \in OFl(2n+1) \times OFl(2n+1) \mid \; M_. \text{ and } F_. \text{ meet in dimensions described by the } \bullet\text{-configuration}\}.$$

$X_\bullet$ is an example of a double Schubert cell. See Fig. 1.

The intersection dimensions of two transverse flags are encoded by black checkers in positions $(i, 2n+2-i)$ for $1 \leq i \leq 2n+1$. We will call this configuration $\bullet_{init}$, the *initial* black checker configuration. See Fig. 2.

[1] In this paper, we use the complex numbers as the base field. It should be noted that more generally, with certain modifications such as replacing the cohomology ring H^{2*} with the Chow ring CH^*, the complex numbers can be replaced with any algebraically closed field of characteristic $\neq 2$.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
M_1	0	0	0	0	0	0	1
M_2	0	0	0	0	0	1	2
M_3	0	0	1	1	1	2	3
M_4	0	0	1	2	2	3	4
M_5	0	0	1	2	3	4	5
M_6	0	1	2	3	4	5	6
M_7	1	2	3	4	5	6	7



	F_1	F_2	F_3	F_4	F_5	F_6	F_7
M_1	0	0	0	0	0	0	1
M_2	0	0	0	0	0	0	1
M_3	0	0	1	1	1	1	2
M_4	0	0	1	1	1	1	2
M_5	0	0	1	1	1	1	2
M_6	0	0	1	1	1	2	3
M_7	0	0	1	1	1	2	3

Fig. 1. The black checkers encode the dimensions of $M_i \cap F_j$ listed on the left and the white checkers encode the dimensions of $V \cap M_i \cap F_j$ listed on the right.

	F_1	F_2	F_3	F_4	F_5
M_1	0	0	0	0	1
M_2	0	0	0	1	2
M_3	0	0	1	2	3
M_4	0	1	2	3	4
M_5	1	2	3	4	5

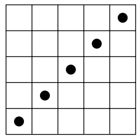


Fig. 2. $\bullet_{init}$-configuration for $n = 2$.

The configuration of black checkers in positions (i, i) for $1 \leq i \leq 2n+1$ describes the diagonal of $OFl(2n+1)\times OFl(2n+1)$. Such a configuration will be called $\bullet_{final}$, the *final* black checker configuration. The corresponding double Schubert cell is $X_{\bullet_{final}}$.

Given an isotropic flag $F_.$ and an isotropic flag $M_.$ that is transverse to $F_.$, we give a sequence of n^2 rational curves in $OFl(2n+1)$, each of degree one or two. The sequence of curves moves the flag $M_.$ until it coincides with $F_.$. Traveling along each curve causes a minimal increase in intersection between $M_.$ and $F_.$, a codimension one degeneration. A degeneration corresponds to moving black checkers on the $(2n+1)\times(2n+1)$ checkerboard. The prescribed sequence of black checker moves is called the *specialization order*.

Theorem 2.1. *There is a sequence of n^2 codimension one degenerations taking an arbitrary isotropic flag to a fixed isotropic flag. Each degeneration respects isotropy and corresponds to a curve of degree one or two in the orthogonal flag variety.*

Consider the subset S of the symmetric group on $2n+1$ elements.

$$S = \{s_0, s_1, \ldots, s_{n-1}\} \subset S_{2n+1}$$

where $s_0 = (n, n+2)$ and $s_i = (n+1+i, n+2+i)(n+1-i, n-i)$ for $1 \leq i \leq n-1$. S generates W, the Weyl group for B_n. Note that $s_i^2 = 1$, $(s_is_{i+1})^3 = 1$ for $1 \leq i \leq n-2$, $(s_0s_1)^4 = 1$, and $(s_is_j)^2 = 1$ for $|i-j| \geq 2$. So (W, S) is a Coxeter system [6]. The specialization order comes from a path through W under the Bruhat order [3], beginning with a representation of the longest word ω_0 (length n^2) and ending with the identity, 1. Let

$$\omega_0 = (s_{n-1}s_{n-2}\ldots s_0\ldots s_{n-2}s_{n-1})(s_{n-2}s_{n-3}\ldots s_0\ldots s_{n-3}s_{n-2})\ldots(s_1s_0s_1)(s_0). \tag{1}$$

If $s_j \in S$ is the rightmost letter of a word ω, apply the permutation s_j to ω, giving $\omega' = \omega s_j$. By the properties of the Coxeter system (W, S), we have $s_j^2 = 1$ so ω' has length 1 less than ω. The representation of ω_0 in equation (1) (reading right to left) gives the specialization order for the deformation of $M_.$ into $F_.$.

An s_0 move swaps the black checkers in rows n and $n+2$. An s_i move, $i \neq 0$, swaps the black checkers in rows $n+1+i$ and $n+2+i$ and simultaneously swaps the checkers in rows $n+1-i$ and $n-i$. See Fig. 3.

We now make precise the degeneration at each step by describing an explicit rational curve that the $M_.$ flag follows to move one step closer to becoming the $F_.$ flag. Given any $F_.$ and $M_.$ in $\bullet$-position, we can choose a basis such that $F_.$ is the standard flag and $M_.$ is given by the $\bullet$-configuration. Specifically, $F_j = \langle e_1, \ldots, e_j\rangle$ and $M_i = \langle e_{j_1}, \ldots, e_{j_i}\rangle$ where j_k is the column of the black checker in row k.

For a fixed $F_.$, we describe a curve in $X_\bullet \cup X_{\bullet_{next}}$ as the set of points $(F_., M_.^p)$ for $p = [s,t] \in \mathbb{P}^1$ with the properties:

1. $M_.^{[0,1]} = M_.$
2. $(F_., M_.^p) \in X_\bullet$ for $p \neq [1, 0]$
3. $(F_., M_.^{[1,0]}) \in X_{\bullet_{next}}$.

The degeneration for an s_i move is linear while the degeneration for an s_0 move is quadratic.

Rational curve for an s_i move. In an s_i move, $i \neq 0$, black checkers in rows $n+1-i$ and $n-i$ swap and black checkers in rows $n+1+i$ and $n+2+i$ swap. While there are four checkers moving, only two spaces, M_{n-i} and $M_{n-i}^\perp = M_{n+1+i}$ are actually changing during the degeneration. Since $M_{n-i}^\perp$ is determined by M_{n-i}, we can describe the curve by showing what happens to the flag

$$M_.^p = (M_1 \subsetneq \cdots \subsetneq M_{n-i}^p \subsetneq M_{n+1-i} \subsetneq \cdots \subsetneq M_n).$$

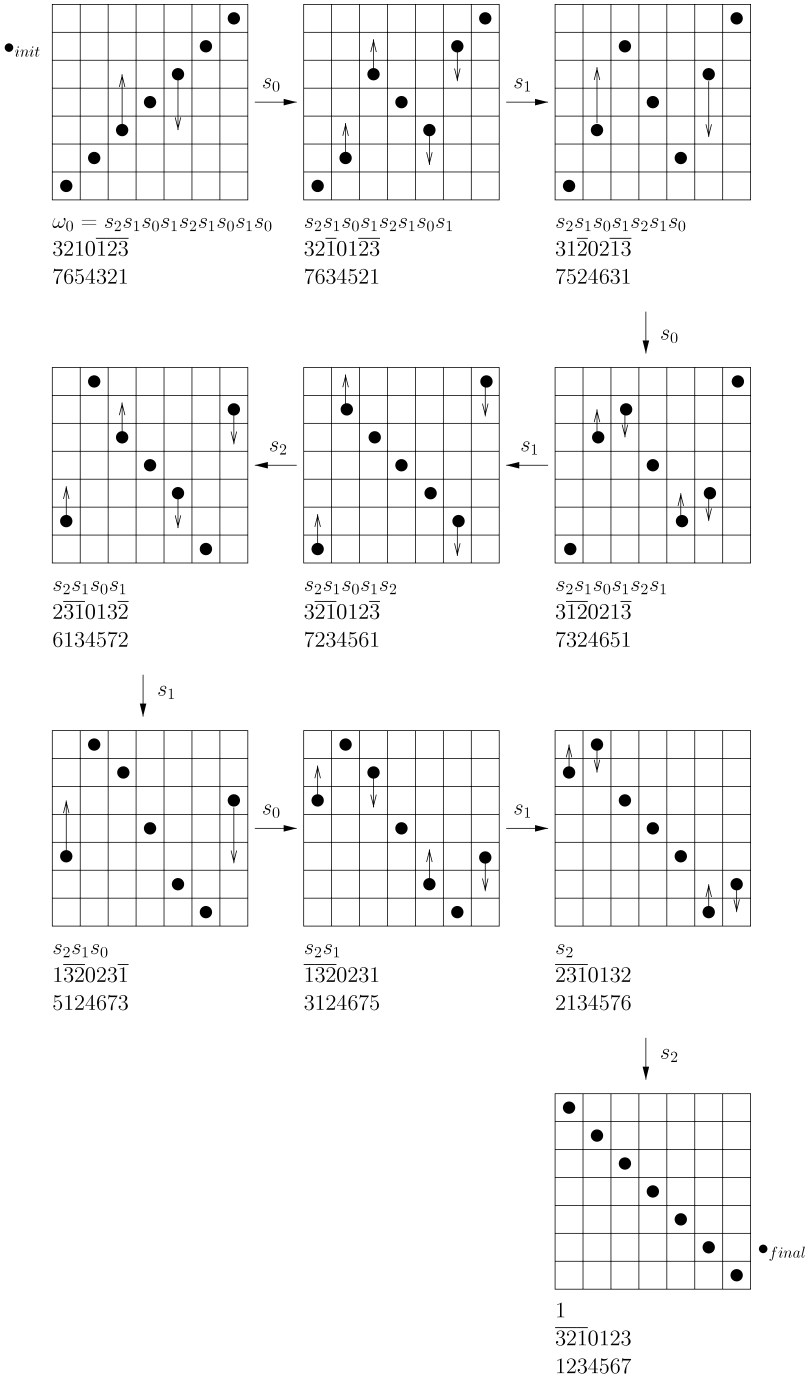


Fig. 3. Specialization order for B_3. Each step is labeled with the corresponding element of the Weyl group, a permutation of $\{\overline{3}, \overline{2}, \overline{1}, 0, 1, 2, 3\}$, and a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$.

For all $p = [s, t] \in \mathbb{P}^1$, we need $M^p_{n-i} \subset M_{n+1-i}$. Define $M^p_{\cdot}$ as

$$M^p_k = M_k = \langle e_{j_1}, \ldots, e_{j_k} \rangle$$

for $1 \le k \le n - i - 1$ and for $n + 1 - i \le k \le n$ and

$$M^p_{n-i} = M_{n-i-1} + \langle s e_{j_{n+1-i}} + t e_{j_{n-i}} \rangle.$$

Note that for any choice $p \in \mathbb{P}^1$, M^p_{n-i} is isotropic and $M^p_{n-i} \subset M_{n+1-i}$.

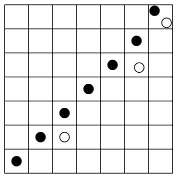


Fig. 4. For $n = 3$, $\tau_{(1)}\tau_{(3,1)}$ corresponds to this initial position.

Rational curve for an s_0 move. For an s_0 move, black checkers in rows n and $n+2$ swap positions. Here, two checkers are moving and two spaces, M_n^p and $M_{n+1}^p = (M_n^p)^\perp$ are moving in the degeneration. Since M_{n+1}^p is determined by M_n^p, we describe the curve by considering $M_\cdot^p = (M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n^p)$. We need M_n^p isotropic and $M_n^p \subset M_{n+2}$. Define $M_\cdot^p$ by

$$M_k^p = \langle e_{j_1}, \ldots, e_{j_k} \rangle$$

for $1 \le k \le n-1$ and

$$M_n^p = M_{n-1} + \langle 2s^2 e_{j_{n+2}} + 2ts e_{n+1} - t^2 e_{j_n} \rangle.$$

Note that $j_{n+1} = n+1$ since the center checker is in position $(n+1, n+1)$.

For any choice $p = [s,t] \in \mathbb{P}^1$, M_n^p is isotropic and quadratic terms are used to satisfy isotropy.

2.2. White checkers

In addition to the black checkers, we place n white checkers on the board. These white checkers encode the dimensions, $\dim(V \cap M_i \cap F_j)$, for $F_\cdot$ and $M_\cdot$ in $\bullet$-position and $V \in OGr(n, 2n+1)$. The number of white checkers weakly northwest of position (i,j) is $\dim(V \cap M_i \cap F_j)$. Note that the southernmost row then encodes $\dim(V \cap F_j)$ and the easternmost column encodes $\dim(V \cap M_i)$. See Fig. 1 for an example.

Definition 2.

$$X_{\circ\bullet} = \{(V, M_\cdot, F_\cdot) \in OGr(n, 2n+1) \times X_\bullet \mid V \text{ meets } M_\cdot \text{ and } F_\cdot \text{ in dimensions described by the } \circ\bullet\text{-configuration}\}$$

Definition 3. A white checker configuration $\circ$ is *happy with respect to* $\bullet$ if each white checker has one black checker weakly north of it in the same column and one black checker weakly west of it in the same row (this is Vakil's definition of happy). When there is no confusion over which $\bullet$-configuration is meant, then we just say *happy*.

Specific to the B_n setting: A white checker configuration $\circ$ is *pairwise happy* if it is happy and if there is a white checker in position (r_i, c_i) and a white checker in position (r_j, c_j), then $r_i + r_j \ne 2n+2$ and $c_i + c_j \ne 2n+2$ for any $1 \le i, j \le n$ (even $i = j$).

And finally, a white checker configuration $\circ$ is *isotropically happy* if it is pairwise happy and there exists some $V \in OGr(n, 2n+1)$ that meets the flags in exactly the way described by the $\circ\bullet$ configuration. In other words, there cannot be a more specialized white checker configuration that describes the full set of isotropic V's that meet the flags in the way described by the original configuration.

Any given $\circ\bullet$-configuration can be determined to be isotropically happy or not by asking about ideal membership in an ideal generated by quadratics. This can (in theory, in any given case) be solved algorithmically by Gröbner basis methods. It is an open question to determine combinatorially if a white checker configuration is isotropically happy.

Initial configurations. For a Schubert question posed as $\overline{\Omega_\lambda}(F_\cdot) \cap \overline{\Omega_\mu}(M_\cdot)$, we describe how to set up the initial white checker configuration: label columns and rows with $\overline{n}, \overline{n-1}, \ldots, \overline{1}, 0, 1, 2, \ldots, n-1, n$ where 0 is the middle column and row. Let $\lambda^\vee$ be the strict partition whose parts complement the parts of λ in the set $\{1, 2, \ldots, n\}$. For $1 \le k \le \ell(\lambda)$, we define

$$j_k = \overline{\lambda_k}$$

and for $1 \le k \le \ell(\lambda^\vee)$, we define

$$j_{\ell(\lambda)+k} = \lambda^\vee_{\ell(\lambda^\vee)+1-k}.$$

For μ, we define $i_1 < i_2 < \cdots < i_n$, similarly.

On a $\bullet_{init}$-configuration, define $\circ_{init}$ as the *initial white checker configuration* for $\overline{\Omega_\lambda}(F_\cdot) \cap \overline{\Omega_\mu}(M_\cdot)$. $\circ_{init}$ is constructed by placing white checkers in positions (i_k, j_{n+1-k}) for $1 \le k \le n$. See Fig. 4 for an example $\circ_{init}$-configuration.

We make some observations about $\circ_{init}$ but omit the proofs.

1. The white checker configuration $\circ_{init}$ is pairwise happy.
2. If μ is not contained in $\lambda^\vee$ then the initial white checker configuration $\circ_{init}$ is not happy and $\Omega_\lambda(F_\cdot) \cap \Omega_\mu(M_\cdot) = \emptyset$.
3. On a $\bullet_{init}$ checkerboard, the $\circ_{init}$ configuration is the least specialized $\circ$-configuration that still describes the required intersections for $\Omega_\lambda(F_\cdot) \cap \Omega_\mu(M_\cdot)$.

Definition 4. A checker diagram is described as *midsort* if the black and white checkers are positioned in such a way that the black checkers are in one of the specialization order configurations and the white checkers are isotropicallly happy and are in positions that follow from prescribed moves beginning with a $\circ_{init}$-configuration.

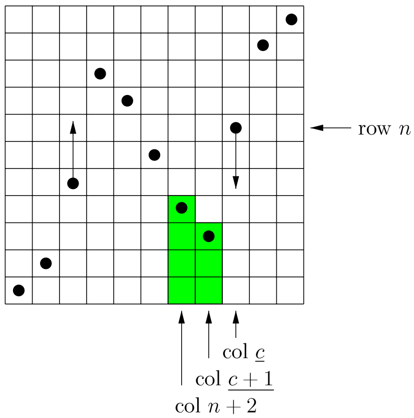


Fig. 5. If there is a white checker in row n in an s_0 move, then look for the northern most white checker in the shaded region ($n+2 \le col \le \underline{c+1}$).

Table 1
White checker moves for the s_0 case when there is no white checker in column $c+1$.

		Is there a white checker in row n?		
		Yes, in $col = \underline{c}$	Yes, in $col > \underline{c}$	No
Top WC in column	Yes	Swap	Swap if no blocker *or* stay	Stay
$n+2 \le col \le \underline{c+1}$?	No	Stay	Stay	Stay

We will use underline notation to denote

$$\underline{k} = 2n + 2 - k. \tag{2}$$

Definition 5. Define c (where $c < n$) as the rightmost column with a black checker in position $(\underline{c}, c)$. In other words, c is the column of the rightmost black checker on the left antidiagonal. Then $\underline{c} > n + 2$ is the column of the (rightmost) descending black checker.

The following characteristics of midsort can be proven by induction when the analysis of all degenerations is complete. We state them here as conjectures.

Conjecture 2. *In a midsort checker diagram, white checkers in columns* $1 \le col \le c$ *and columns* $\underline{c} \le col \le 2n + 1$ *decrease in rows from west to east.*

Conjecture 3. *In a midsort checker diagram, white checkers in columns* $c < col < \underline{c}$ *increase in rows from west to east.*

Reading the final answer. A final $\circ\bullet$-configuration has black checkers in positions (k, k) for $\overline{n} \le k \le n$. The n white checkers are in positions along the same diagonal. We determine ν for $\Omega_\nu(F.)$ by recording the positions of the white checkers in columns (or equivalently rows) $\overline{n} \le col < 0$. Call these positions $\alpha_1 < \alpha_2 < \cdots < \alpha_{\ell(\nu)} < 0$. Then $\nu = (\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_{\ell(\nu)}})$.

3. Combinatorial statement

Given a midsort checker configuration for an s_0 move, we give a combinatorial statement that describes geometrically the outcome of the degeneration.

Rule for s_0 moves. For an s_0 move with no white checker in column $c + 1$, the following is the combinatorial rule: There is either a white checker in row n or in row $n + 2$, but not both. If there is not a white checker in row n, then we call this the trivial case and the white checkers stay. If there is a white checker in row n, the row of the descending black checker, then we consider columns $n + 2$ through $\underline{c+1}$. See Fig. 5. If there is a white checker in one of these columns, we choose the top most white checker. The location of this white checker and the white checker in row n determine if these checkers stay, swap, or stay and swap (in a split). In the split possibility, the pair of checkers can stay, or if there are no white checkers in the rectangle between them, they can swap. A white checker in the rectangle is called a *blocker*. See Fig. 6 for an example. Table 1 summarizes the s_0 white checker moves (when there is no white checker in column $c + 1$). The case of a white checker on column $c + 1$ is left open.

4. Proofs for quadratic degenerations

The overall strategy here is the same as developed in [14]. In the trivial case, when there is no white checker in row n, we consider the closure of $X_{\circ\bullet}$ in $OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$, and show that the divisor is $X_{\circ_{stay}\bullet_{next}}$. After the trivial case, we will use the orthogonal Bott–Samelson variety. We define the orthogonal Bott–Samelson variety, $OBS(Q_\circ)$, as the subvariety of $BS(Q_\circ)$ where $V_{\max}$ is isotropic. See [14] for a treatment of $BS(Q_\circ)$, the Bott–Samelson variety as it applies to checker

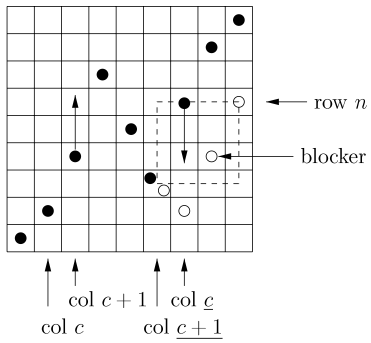


Fig. 6. Example of a blocker in an s_0 move.

diagrams. Instead of considering the divisor D on the closure of $X_{\circ\bullet}$ in $OGr(n, 2n+1) \times (X_\bullet \cup X_{\bullet_{next}})$, we consider the corresponding divisor D_Q on the closure of $OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$. See diagram (3). The map π is the projection from a point $(V., M., F.) \in Cl_{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet}$ to $(V, M., F.) \in Cl_{OGr(n,2n+1)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet}$ that drops all subspaces associated to the quilt except $V_{\max}$.

$$\begin{tikzcd}
Cl_{OBS(Q_\circ)\times X_\bullet}X_{\circ\bullet} \arrow[r, hook, "open"] \arrow[d] & Cl_{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet} \arrow[d, "\pi"'] & D_Q \arrow[l, hook', "closed"'] \arrow[d] \\
Cl_{OGr(n,2n+1)\times X_\bullet}X_{\circ\bullet} \arrow[r, hook, "open"] \arrow[d] & Cl_{OGr(n,2n+1)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet} \arrow[d] & D \arrow[l, hook', "closed"'] \arrow[d] \\
X_\bullet \arrow[r, hook, "open"] & X_\bullet \cup X_{\bullet_{next}} & X_{\bullet_{next}} \arrow[l, hook', "closed"']
\end{tikzcd} \tag{3}$$

4.1. Trivial case

The trivial case for s_0 moves occurs when there is no white checker in row n.

Theorem 4.1. *For an s_0 move, if there is no white checker in row n then $D = X_{\circ_{stay}\bullet_{next}}$.*

Proof. Let $X'_{\circ\bullet}$ be the image of the projection

$$p_n : X_{\circ\bullet} \to OGr(n, 2n+1) \times OFl(\hat{n}) \times OFl(2n+1)$$

by forgetting M_n and $M_n^\perp = M_{n+1}$. For a general point $p_n(t) \in p_n(X_{\circ\bullet})$, the fiber $p_n^{-1}(t)$ is isomorphic to an open subset of $Y = O\mathbb{P}(M_{n+2}/M_{n-1}) \subset \mathbb{P}(M_{n+2}/M_{n-1})$. The closure of this set in $OGr(n, 2n+1) \times (X_\bullet \cup X_{\bullet_{next}})$ is all of Y. Since there are no white checkers in row n or $n+1$, there are no restrictions coming from $V \in OGr(n, 2n+1)$ on the choice of M_n.

Consider the section s of p_n, defined as follows: for the point

$$p_n(t) = (V, M_1 \subset \cdots \subset M_{n-1}, F.)$$

we need M_n such that

$$M_{n-1} \subset M_n \subset M_{n-1} + (M_{n+2} \cap F_{c+1}^\perp)$$

where $\dim(M_n) = n$ and M_n is isotropic. Let $L = M_{n+2} \cap F_{c+1}$. For any $(M., F.) \in X_\bullet$, L has dimension 1. Define $M_n = M_{n-1} + L$. This choice of M_n gives us a point in $OGr(n, 2n+1) \times X_{\bullet_{next}}$:

$$\begin{aligned}
(M., F.) \in X_{\bullet_{next}} &\iff \dim(M_n \cap F_c^\perp) = \dim(M_n \cap F_{c+1}^\perp) \\
&\iff \dim((M_{n-1} + L) \cap F_c^\perp) = \dim((M_{n-1} + L) \cap F_{c+1}^\perp)
\end{aligned}$$

which are equal since $L \subset F_{c+1} \subset F_{c+1}^\perp \subset F_c^\perp$ and $M_{n-1} \cap F_c^\perp = M_{n-1} \cap F_{c+1}^\perp$ in the $\bullet$-configuration. We now show that this choice for M_n is the unique choice such that $(M., F.) \in X_{\bullet_{next}}$ instead of in $X_\bullet$. By the black checker configuration, we have

$$M_{n+1} \cap F_c^\perp = M_{n+1} \cap F_{c+1}^\perp \iff (M., F.) \in X_{\bullet_{next}}.$$

Now, $M_{n+1} \cap F_c^{\perp} = (M_n + F_c)^{\perp}$ and $M_{n+1} \cap F_{c+1}^{\perp} = (M_n \cap F_{c+1})^{\perp}$. So

$$\begin{aligned}(M_., F_.) \in X_{\bullet_{next}} &\iff M_n + F_c = M_n + F_{c+1}\\ &\iff \dim(M_n \cap F_c) = \dim(M_n \cap F_{c+1}) - 1\\ &\iff \text{there is a line } \mathcal{L} \subset F_{c+1} \text{ such that } M_n = M_{n-1} + \mathcal{L}.\end{aligned}$$

And $M_n \subset M_{n+2}$ means $\mathcal{L} \subset M_{n+2}$. So $\mathcal{L} \subset M_{n+2} \cap F_{c+1}$. Now, $\dim(M_{n+2} \cap F_{c+1}) = 1$, so $\mathcal{L}$ is unique and $\mathcal{L} = L$. We've shown there is a unique divisor D in $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ that satisfies the conditions for $X_{\bullet_{next}}$.

We now show that the divisor D has multiplicity 1. We give a test family $\mathcal{F}$ through a general point $t \in (V, M_., F_.)$ of $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ meeting the divisor $D = X_{\circ_{stay}\bullet_{next}}$ with multiplicity 1. For our general point of $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$, we know $(M_., F_.) \in X_\bullet$. We choose a basis for $F_.$ and $M_.$:

- Let $F_.$ have the standard basis with $F_j = \langle e_1, \ldots, e_j\rangle$.
- Let $M_.$ have the basis that depends on the $\bullet$-configuration. In particular,

  $$M_{n-1} = \langle e_{2n+1}, e_{2n}, \ldots, e_{2n+2-c}; e_{c+2}, e_{c+3}, \ldots, e_n\rangle$$

  and

  $$M_{n+2} = M_{n-1} + \langle e_{\underline{c}}, e_{n+1}, e_{c+1}\rangle.$$

Build the one-dimensional test family $\mathcal{F} = \{(V', M'_., F'_.)\}$ as follows:

- Let $V' = V$
- Let $F'_. = F_.$
- For $1 \le i \le n-1$, let $M'_i = M_i$ and $(M'_i)^{\perp} = M_i^{\perp}$
- This leaves M'_n and $M'_{n+1} = (M'_n)^{\perp}$. Define

  $$M'_n = \left\langle M_{n-1}, -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1}\right\rangle$$

  for $[s, t] \in \mathbb{P}^1$. M'_n is isotropic because

  $$\left\langle M_{n-1}, -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1}\right\rangle \subset M_{n+2} = M_{n-1}^{\perp}$$

  and $\mathcal{L} = \langle -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1}\rangle$ is itself isotropic.

We define the family $\mathcal{F}$ to be the the open subset of $\{(V', M'_., F'_.)\}$ described above where $t \neq 0$. When $s \neq 0$ then $(M'_., F'_.) \in X_\bullet$ so $\mathcal{F} \not\subset D$. And when $[s, t] = [0, 1]$ then $(M'_., F'_.) \in X_{\bullet_{next}}$. So $\mathcal{F}$ meets D.

The divisor D on $\mathcal{F}$ is given by

$$\begin{aligned}M'_n \cap F_c^{\perp} \subset F_{c+1}^{\perp} &\iff \dim(M'_n \cap F_{c+1}) = 1\\ &\iff \dim((M_{n-1} + \mathcal{L}) \cap F_{c+1}) = 1.\end{aligned}$$

Now, in both $X_\bullet$ and $X_{\bullet_{next}}$ we have that $\dim(M_{n-1} \cap F_{c+1}) = 0$, so $\dim((M_{n-1} + \mathcal{L}) \cap F_{c+1}) = 1 \iff \mathcal{L} \subset F_{c+1}$. This is equivalent to

$$\left\langle -\frac{1}{2}s^2 e_{\underline{c}} + s e_{n+1} + e_{c+1}\right\rangle \subset F_{c+1},$$

which is true if and only if $-\frac{1}{2}s^2 = 0$ and $s = 0$, a multiplicity one condition. So $\mathcal{F}$ meets D with multiplicity one and thus D has multiplicity one. $\square$

4.2. Nontrivial case

Similar to [14], we describe a closed subscheme of $OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$ and show it is $Cl_{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ (this is Theorem 4.3). The subscheme will be constructed as the intersection of two subvarieties of an open subset of a tower of projective and quadric bundles over $OBS(Q_\circ)$.

A note on abusive notation: We say $X_{\circ\bullet} \subset OBS(Q_\circ) \times X_\bullet$ when in fact, $X_{\circ\bullet}$ is a subset of $OGr(n, 2n+1) \times X_\bullet$. There is however, a natural injection

$$X_{\circ\bullet} \hookrightarrow OBS(Q_\circ) \times X_\bullet$$

which takes $(V, M_., F_.) \mapsto ((V_m)_{m\in Q_\circ}, M_., F_.)$ such that for $m \in Q_\circ$ in position (i, j) on the checker board, $V_m = V \cap M_i \cap F_j$. So without further note, we say $X_{\circ\bullet} \subset OBS(Q_\circ) \times X_\bullet$ and leave the injection implicit.

Definition 6. Let $\mathbf{m}(M_i)$ $(1 \le i \le 2n+1)$ be the maximum element $\mathbf{m}$ of $Q_\circ$ in rows up through i. Define $\mathbf{m}(F_j)$ similarly to be the maximum element $\mathbf{m} \in Q_\circ$ in columns up through j. In particular, we define $\mathbf{a} = \mathbf{m}(F_{\underline{c+1}})$ (c is defined in Definition 5), an element we will reference often.

We've already dealt with the trivial cases so we will assume for the remainder of the discussion that there is a white checker in row n. We assume also that there is no white checker in column $c+1$. A white checker in columns $c+1$ may require a different approach and is left open.

Consider a subspace

$$T \subset OBS(Q_\circ) \times OFl(2n+1) \times OFl(1, \dots, c, 2n+1).$$

We describe how to build T and discuss some of its properties. We will then define spaces Q, $W_\circ$, and $W_{\bullet\bullet_{next}}$ which are fibered over T. The spaces T, Q, $W_\circ$, and $W_{\bullet\bullet_{next}}$ are defined for any degeneration in the specialization order (linear or quadratic). Theorem 4.2, Definitions 6–12, and Lemma 1 are stated generally for both linear and quadratic degenerations. Theorems 4.3–4.6 are proved for quadratic degenerations only.

4.3. Building T

The reader may find it helpful to reference Figs. 9 and 10, or 12 during this section. Start with the base space $OBS(Q_\circ)$. For a point $(V_\alpha)_{\alpha\in Q_\circ} \in OBS(Q_\circ)$, build $M_.$ in the following way "from outside to inside." Let $M_0 = \langle 0\rangle$, then for $1 \le i \le n$, choose M_i such that

1. $M_{i-1} \subset M_i \subset M_{i-1}^\perp$
2. M_i is isotropic
3. $V_{m(M_i)} \subset M_i \subset V_{m(M_{\underline{i}})}^\perp$.

Complete the isotropic flag $M_.$ by defining for $0 \le i \le n$, $M_{2n+1-i} = M_i^\perp$.

For a point $((V_., M_.)$, build the partial isotropic flag $F_{\le c}$ in a similar way to $M_.$. Let $F_0 = \langle 0\rangle$, then for $1 \le j \le c$, choose F_j such that

1. $F_{j-1} \subset F_j \subset F_{j-1}^\perp$
2. F_j is isotropic
3. $V_{m(F_j)} \subset F_j \subset V_{m(F_{\underline{j}})}^\perp$
4. F_j is transverse to the flag $M_.$

Then for $0 \le j \le c$ define $F_{2n+1-j} = F_j^\perp$. This completes the space T.

Theorem 4.2. *At each midsort step in the degeneration, the space T is reduced and irreducible.*

Proof. We build T on $OBS(Q_\circ)$ by choosing $M_1, M_2, \dots, M_n$ and then $F_1, \dots, F_c$. If there is no white checker in row $i+1$ then choosing M_{i+1} is equivalent to choosing an isotropic line in $(M_i^\perp \cap V_{m(M_{\underline{i+1}})}^\perp)/M_i$. If there is a white checker in row $i+1$ then there is exactly one choice for M_{i+1}, namely $M_{i+1} = M_i + V_{m(M_{i+1})}$. We choose the F_j's in a similar way with the additional open condition that F_j is transverse to the $M_.$ flag.

We'd like to show that T is reduced and irreducible. Together these amount to showing that at each step where we add M_{i+1} when there is no white checker in row $i+1$, that the rank of the symmetric bilinear form, $rank(B)$, on $(M_i^\perp \cap V_{m(M_{\underline{i+1}})}^\perp)/M_i$ is greater than or equal to 3.

Let $W = V_{m(M_{\underline{i+1}})}^\perp$, $\dim(W) = k$, $rank(B|_W) = r$ where r is odd. Also let $V = W^\perp = V_{m(M_{\underline{i+1}})}$ and $M = M_i$. Note that $V_{m(M_{\underline{i+1}})} \subset V_{m(M_{\underline{i}})} \subset M_i^\perp$ so $M \subset W$. Our claim is now reworded: $rank(B)$ on $(M^\perp \cap W)/M$ is greater than or equal to 3.

We do a change of basis so that our form B is nicer looking. See Fig. 7. r is odd so let $r = 2q+1$. Then the center 1 of the $r \times r$ block (call this block R) of the matrix is in position $(q+1, q+1)$. M is isotropic so the block M cannot meet the antidiagonal of ones in block R. Thus we must have $i+q+1 \le k$. This implies that $i < k-q$.

We have two cases to consider depending whether blocks R and M overlap or not.

Case 1

They do not overlap, i.e. $i \le k-r$. Then $M_i^\perp = W$ and rank $((M_i^\perp \cap W)/M_i) = r$. Since r is odd, either $r = 1$ or $r \ge 3$. If $r = 1$ then $V_{m(M_{\underline{i+1}})}$ is maximal isotropic. This is because $W = V_{m(M_{\underline{i+1}})}^\perp$ is the perp of an isotropic space and if $rank(B|_W) = 1$, then we've only added the middle vector so $V_{m(M_{\underline{i+1}})}$ must have been dimension n. This means there are n white checkers in rows $1, \dots, \underline{i+1}$. If there are n white checkers in rows $1, \dots, \underline{i+1}$ then we must be in the maximal case. And if we are in the maximal case and we've assumed there is no white checker in row $i+1$, then there must be a white checker in row $2n+2-(i+1) = 2n+1-(i+1)+1 = \underline{i+1}+1$. So there cannot be n white checkers in rows $1, \dots, \underline{i+1}$. Thus $r \ne 1$, which implies $r \ge 3$.

Case 2

Blocks R and M overlap, i.e. $i > k-r$. See Fig. 8. M and $M^\perp$ are spanned by the basis vectors

$$M = \langle e_k, e_{k-1}, \dots, e_{k-i+1}\rangle$$
$$M^\perp = \langle e_k, \dots, e_{k-i+1}, e_{k-i}, \dots, e_{r+1-(k-i)}\rangle$$

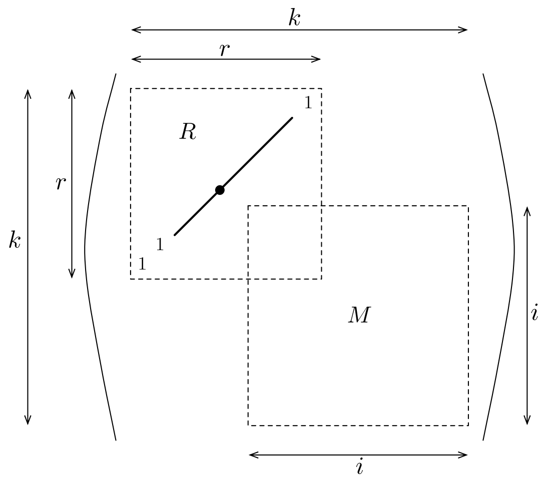


Fig. 7. The form for the change of basis matrix.

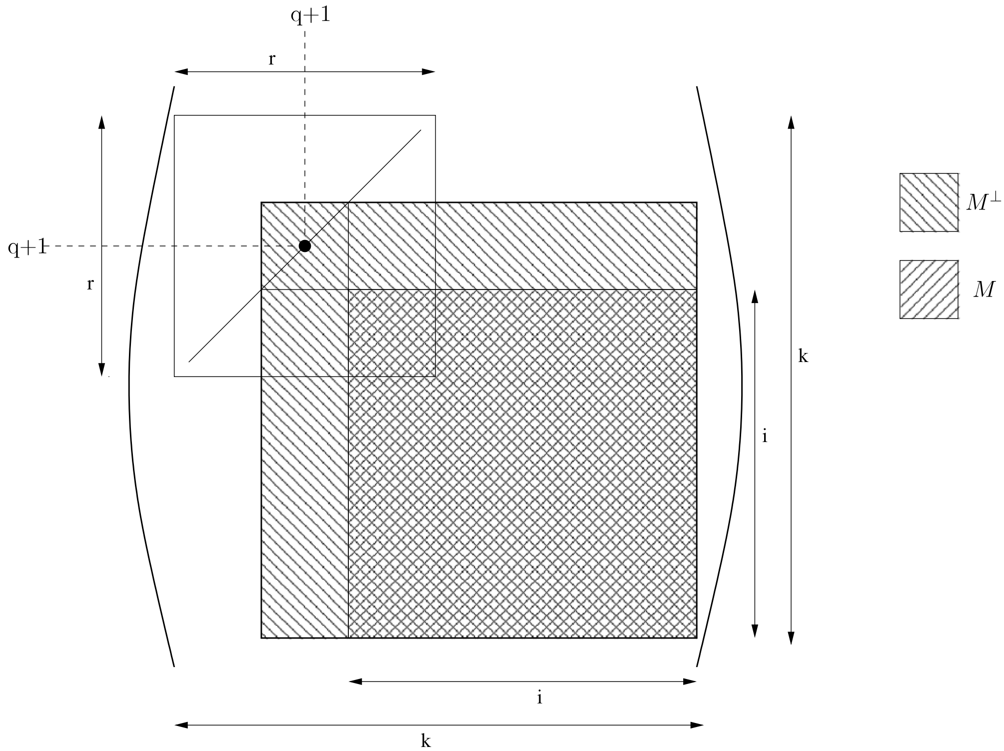


Fig. 8. This is the form of the change of basis matrix if M and R blocks overlap, i.e. $i > k - r$.

so in the quotient space we have

$$(M^\perp \cap W)/M = \langle \bar{e}_{r+1-(k-i)}, \ldots, \bar{e}_{k-i} \rangle$$

and

$$\begin{aligned} rank(\bar{B}) &= (k-i) - (r + 1 - (k-i)) + 1 \\ &= 2(k-i) - r. \end{aligned}$$

We know $r = 2q + 1$ and by hypothesis we have both $i > k - r$ and $i < k - q$. This implies $2(k - i) - r \geq 0$. And $2(k - i) - r$ is odd because r is odd and $2(k - i)$ is even. We again consider the two possibilities $2(k - i) - r = 1$ and $2(k - i) - r \geq 3$.

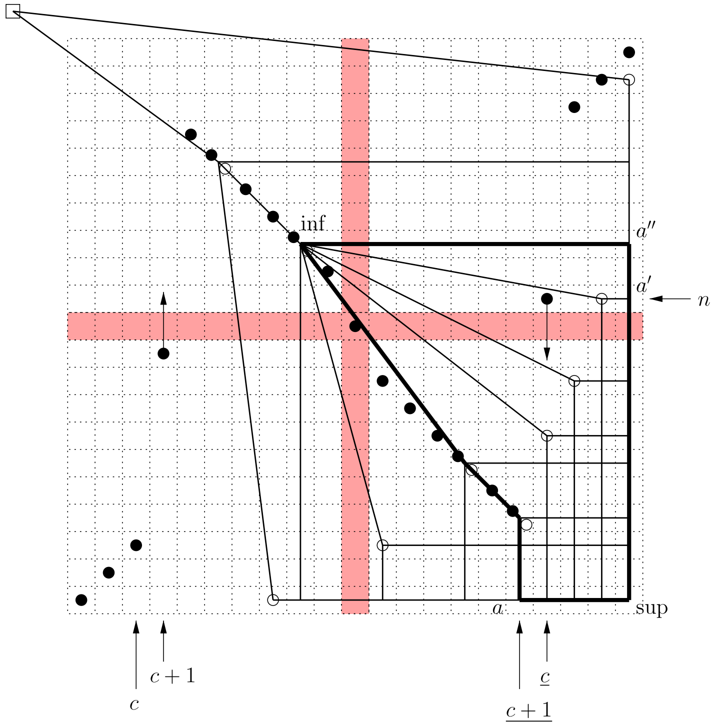


Fig. 9. Example of an s_0 move where there is no white checker in column $c+1$ and there is a white checker in row n. $\inf(a, a'') \neq a, a''$.

Suppose $2(k-i)-r=1$. Then we have

$$\begin{aligned}
&2(k-i)-r=1\\
&2(k-i)=r+1\\
&2(k-i)=(2q+1)+1\\
&k-i=q+1.
\end{aligned}$$

This means the upper left corner of the M block is at position $(q+2, q+2)$.

Let $i(m)$ be the number of white checkers in rows $1, \ldots, m$ of the checker board. Then

$$\begin{aligned}
k &= \dim W\\
&= 2n+1-\dim(V_{m(M_{\underline{i+1}})})\\
&= 2n+1-i(\underline{i+1}).
\end{aligned}$$

And $r = 2n+1-2i(\underline{i+1})$. This is because we have 1 contributing to the rank from the middle row and then we have $2n$ possible more to contribute to the rank r. For each row with a white checker, we do not have that basis vector nor do we have its mirror pair contributing to the rank, a total non-contribution of $2i(\underline{i+1})$. Then noting that $r = 2q+1$, we can conclude that $q = n - i(\underline{i+1})$. From the following calculation, we get that $n = i$.

$$\begin{aligned}
&k-i=q+1\\
&2n+1-i(\underline{i+1})-i=n-i(\underline{i+1})+1\\
&n=i.
\end{aligned}$$

But $i < n$ (recall that we're finding M_{i+1} which is at largest M_n so $i+1 \leq n$ and thus $i < n$) so we have a contradiction. Thus $rank(\overline{B}) = 2(k-i) - r \neq 1$ so $rank(\overline{B}) \geq 3$. $\square$

Spaces built on T. Let $\inf \in Q_\circ$. inf is an important element of $Q_\circ$ for the proof of Theorem 4.3. We will define precisely which element of $Q_\circ$ is named inf separately in the proofs of Theorems 4.3 and 4.4.

Definition 7. For a fixed point $t = ((V_m)_{\mathbf{m} \in Q_\circ}, M_\cdot, F_{\leq c}) \in T$, choose F_{c+1} such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{\inf} \subset F_{c+1}^\perp$.

For fixed $t \in T$ let Q_t be the set of all such F_{c+1}.

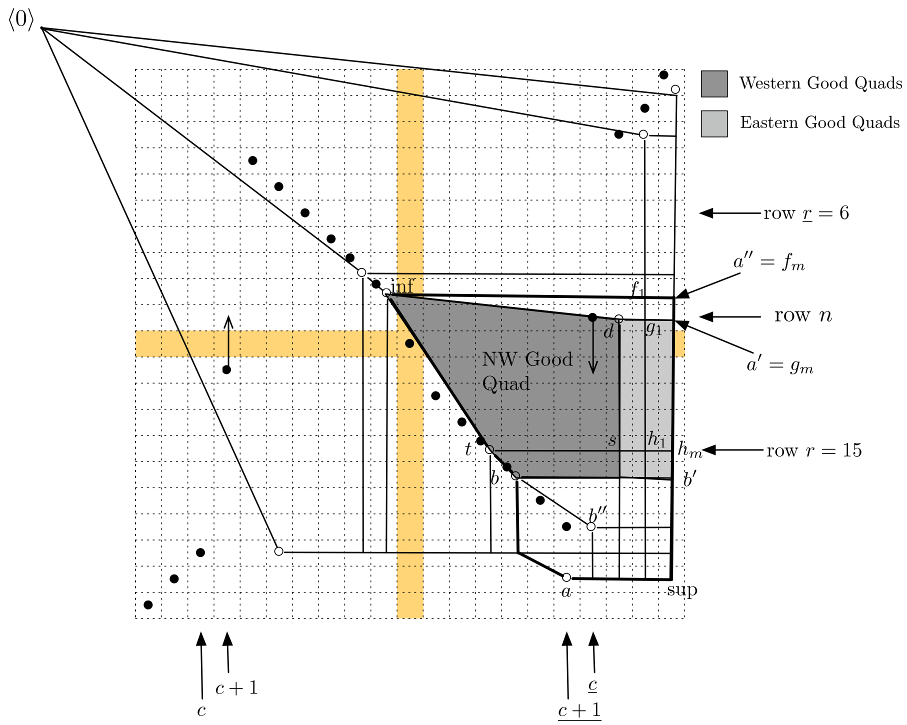


Fig. 10. Example of a nontrivial s_0 move. Labels r, s, t, f_i, g_i, and h_i are discussed in the proof of Theorem 4.6(b).

Note that if $F_c \subset F_{c+1}$ and F_{c+1} is isotropic then the condition $F_{c+1} \subset F_c^{\perp}$ is met automatically.

Definition 8. $Q = \{(t, F_{c+1})|t \in T, F_{c+1} \in Q_t\}$. Q is fibered over T with fibers Q_t.

Definition 9. For a fixed point $t = ((V_m)_{\mathbf{m}\in Q_\circ}, M_., F_{\leq c}) \in T$, choose F_{c+1} such that

1. $(t, F_{c+1}) \in Q_t$
2. $V_{m(F_{c+1})} \subset F_{c+1} \subset V_{\mathbf{a}}^{\perp}$ (see Definition 6).

For fixed $t \in T$ let S_t be the set of all such F_{c+1}.

Definition 10. $W_\circ = \{(t, F_{c+1})|t \in T, F_{c+1} \in S_t\}$. $W_\circ$ is fibered over T with fibers S_t.

Definition 11. For a fixed point $t = ((V_m)_{\mathbf{m}\in Q_\circ}, M_., F_{\leq c}) \in T$, choose F_{c+1} such that

1. $(t, F_{c+1}) \in Q_t$
2. $(F_c^{\perp} \cap M_{d_E-1}) \subset F_{c+1}^{\perp}$
3. $(F_c^{\perp} \cap M_{d_E+1}) \not\subset F_{c+1}^{\perp}$ (this is an open condition).

For fixed $t \in T$ let R_t be the set of all such F_{c+1}.

Note: Here we use d_E to represent the row the eastern descending black checker. In linear degenerations, this row varies. For quadratic degenerations, $d_E = n$.

Definition 12. $W_{\bullet\bullet_{next}} = \{(t, F_{c+1})|t \in T, F_{c+1} \in R_t\}$. $W_{\bullet\bullet_{next}}$ is fibered over T with fibers R_t.

Lemma 1. *$W_{\bullet\bullet_{next}}$ has a natural immersion*

$$W_{\bullet\bullet_{next}} \hookrightarrow OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}}).$$

Proof. Given a point $((V_m)_{\mathbf{m}\in Q_\circ}, M_., F_{\leq c+1}) \in W_{\bullet\bullet_{next}}$, complete the $F_{\leq(c+1)}$ flag as follows: For F_j, $c+2 \leq j \leq n$, column j has a black checker in row r_j. Let $F_j = (M_{r_j} \cap F_{c+1}^{\perp}) + F_{c+1}$. Note that

1. F_j is isotropic. This is because F_{c+1} is isotropic and $F_{c+1}^{\perp} \cap M_{r_j}$ is isotropic because M_{r_j} is isotropic since $r_j \leq n$ for $c+2 \leq j \leq n$. In addition, $(F_{c+1}^{\perp} \cap M_{r_j}) \subset F_{c+1}^{\perp}$ so every vector in $F_{c+1}^{\perp} \cap M_{r_j}$ is orthogonal to every vector in F_{c+1}.

2. F_j has dimension j. There are two cases:
   (a) $r_j < r_{c+1}$. Then $\dim(F_{c+1}^{\perp} \cap M_{r_j}) = j - (c+1)$ and $(F_{c+1}^{\perp} \cap M_{r_j}) \cap F_{c+1} = \langle 0 \rangle$ so

$$\begin{aligned}\dim(F_j) &= \dim(F_{c+1}^{\perp} \cap M_{r_j}) + \dim(F_{c+1}) - \dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1})\\ &= j-(c+1)+(c+1)-0\\ &= j.\end{aligned}$$

   (b) $r_j > r_{c+1}$. Then $\dim(F_{c+1}^{\perp} \cap M_{r_j}) = j - c$ since the black checker in column $c+1$ is now included in the dimension count. This also means $\dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1}) = 1$. So

$$\begin{aligned}\dim(F_j) &= \dim(F_{c+1}^{\perp} \cap M_{r_j}) + \dim(F_{c+1}) - \dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1})\\ &= (j-c)+(c+1)-1\\ &= j. \quad \square\end{aligned}$$

4.4. Quadratic-specific results

We now show that $Cl_{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ is the intersection of the subvarieties $W_\circ$ and $W_{\bullet\bullet_{next}}$.

Theorem 4.3. *We have the scheme-theoretic equality*

$$W_\circ \cap W_{\bullet\bullet_{next}} = Cl_{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}.$$

The strategy of the proof is this: We fix an irreducible component Z of $W_\circ \cap W_{\bullet\bullet_{next}}$ and describe an open subscheme of Z explicitly as a tower of projective and quadric bundles over the dense open subset $OBS(Q_\circ)_\emptyset$. Through a description of the open subscheme of Z as a tower of bundles, we will show that Z has the expected codimension. Note that $W_\circ \cap W_{\bullet\bullet_{next}}$ is irreducible because T is irreducible and the fibers over $OBS(Q_\circ)_\emptyset$ are equidimensional. Since $W_\circ \cap W_{\bullet\bullet_{next}}$ is irreducible and, as we will show, Z is a component of the same dimension, we get that Z must be unique and thus $Z = W_\circ \cap W_{\bullet\bullet_{next}}$.

The following definition will play a large role in the proof of Theorem 4.3.

Definition 13. The *expected codimension* over Q of $W_\circ \cap W_{\bullet\bullet_{next}}$ is

$$\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \operatorname{codim}_Q(W_\circ) + \operatorname{codim}_Q(W_{\bullet\bullet_{next}}).$$

We pause here to also state the result from [5]. This result plays a critical role in the proof of Theorem 4.3.

Lemma 2. *For $k \le n$, let $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ be a k-tuple of integers with $1 \le a_1 < \cdots < a_k \le 2n+1$ with the convention that $a_0 = 0$ and $a_{k+1} = \infty$. Let $Y_{\mathbf{a}}$ be a Schubert variety whose elements are $(V_., M_.)$ where $V_. \in OFl(1, 2, \ldots, k, 2n+1)$ is a partial isotropic flag and $M_. \in OFl(2n+1)$ is a complete isotropic flag (if $k = n$ then $V_.$ is also complete). The variety $Y_{\mathbf{a}}$ is defined as*

$$Y_{\mathbf{a}} = \{(V_., M_.) \mid V_{i(\alpha)} \subset M_\alpha \subset V_{i(\alpha)}^{\perp}\} \subset OFl(1, \ldots, k, 2n+1) \times OFl(2n+1) \tag{4}$$

where $1 \le \alpha \le n$ and $i(\alpha) = \max\{j \mid a_j \le \alpha\}$. Given $\delta \le k$ and $\ell_2 \ge 0$ and integers j and R such that $a_j \le R < a_{j+1}$, let P be the subvariety of $Y_{\mathbf{a}}$ where $\dim(V_\delta \cap M_R) = j + \ell_2$, then $\operatorname{codim}_{Y_{\mathbf{a}}}(P) \ge \ell_2$. Furthermore, if equality holds, then one of the following is true:

1. $\ell_2 = 0$
2. $\ell_2 = 1$, $R \ge n+1$, $a_j < R$, $a_{j+1} = R+1$, *and* $V_{j+1} \subset M_R$ *for all points of* P
3. $\ell_2 = 1$, $R < n$, $a_j < R$, $a_{j+1} = R+1$, *and* $V_{j+1} \subset M_R$ *for all points of* P
4. $\ell_2 = 1$, $R = n$, $a_j < R$, $a_{j+1} = n+2$, *and* $V_{j+1} \subset M_R$ *for all points of* P.

The rest of this section deals specifically with quadratic degenerations, the s_0 moves.

Proof of Theorem 4.3 for s_0 moves. For a nontrivial s_0 move, there is a white checker in row n. Let $\inf = \inf(a, a'')$, $\mathbf{a} = \mathbf{m}(F_{c+1}^{\perp})$, $\mathbf{a}' = \mathbf{m}(M_n)$, $\mathbf{a}'' = \mathbf{m}(M_{n-1})$, and $\sup = \sup(a, a')$. Figs. 9, 10 and 12 are examples of nontrivial s_0 moves. We have three cases.

Case (**i**) $\inf = \mathbf{a}$. If there are no white checkers in columns $n+2 \le col \le \underline{c+1}$ then $\inf = \mathbf{a}$. No white checkers in this region implies that $V_{\mathbf{a}} \subset V_{\inf} + F_c$ which means $V_{\inf}^{\perp} \cap F_c^{\perp} \subset V_{\mathbf{a}}^{\perp}$. We already know that all elements of Q satisfy the conditions $F_{c+1} \subset V_{\inf}^{\perp}$ and $F_{c+1} \subset F_c^{\perp}$, so $F_{c+1} \subset V_{\mathbf{a}}^{\perp}$ is not a new condition. Thus $W_\circ = Q$ and we have $Z = W_{\bullet\bullet_{next}}$. So Z is unique and $\operatorname{codim}_Q Z = \operatorname{codim}(W_{\bullet\bullet_{next}}) = \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$.

Case (**ii**) $\inf \ne \mathbf{a}, \mathbf{a}''$. See Figs. 9 and 10. There must be at least one white checker in columns $n+2 \le col \le \underline{c+1}$, and at least one white checker in the region bounded by $1 \le row < n$ and $\underline{c} \le col \le 2n+1$. We will construct a dense open subscheme of Z. Let Z_V be the image of Z in $OBS(Q_\circ)$, Z_M be the image of Z in $OBS(Q_\circ) \times \{M_.\}$, and Z_F be the image of Z in

$T \subset OBS(Q_\circ) \times \{M.\} \times \{F_{\leq c}\}$. See diagram (5).

$$
\begin{tikzcd}
Q \arrow[d, "\pi_F"'] & W_\circ \cap W_{\bullet\bullet_{next}} \arrow[l, hook'] \arrow[dl] & & \xi_t \arrow[ll] & p_F^{-1}(t) \arrow[l, hook', "codim = \ell_6"] \arrow[r] & Z \arrow[d, "p_F"] \\
T \arrow[d, "\pi_M"'] & & \pi_M^{-1}(V., M.) \arrow[ll] & & p_M^{-1}(V., M.) \arrow[ll, hook', "codim = \ell_5"] \arrow[r] & Z_F \arrow[d, "p_M"] \\
\{(V., M.)\} \arrow[d, "\pi_V"'] & & \pi_V^{-1}(V.) \arrow[ll] & & p_V^{-1}(V.) \arrow[ll, hook', "codim = \ell_4"] \arrow[r] & Z_M \arrow[d, "p_V"] \\
OBS(Q_\circ) & & & & & Z_V \arrow[lllll, hook', "codim = \ell_1"]
\end{tikzcd}
\tag{5}
$$

Let $\boxed{\ell_1} = \operatorname{codim}_{OBS(Q_\circ)} Z_V$. Z_V is contained in some closed stratum of codimension at most ℓ_1 which corresponds to a set S of at most ℓ_1 quadrilaterals of $Q_\circ$ (strata are discussed in [14]). Thus $\ell_1 \geq |S|$. If $|S| = \ell_1$ then Z_V *is* the stratum $OBS(Q_\circ)_S$.

We next consider the choices for $M.$ with the conditions described in the construction of T. Let $\boxed{\ell_4}$ be the codimension of Z_M in the fibration

$$\pi_V^{-1}(Z_V) \to Z_V.$$

Define for a general point of Z,

$$
\begin{aligned}
\boxed{\ell_2} &= \dim(V_{\sup} \cap M_{n-1}) - \dim(V_{m(M_{n-1})}) \\
&= \dim(V_{\sup} \cap M_{n-1}) - \dim(V_{\mathbf{a}''}).
\end{aligned}
$$

By Lemma 2, taking $R = n-1$, $j = \dim(m(M_{n-1})) = \dim(a'')$, $\delta = \dim(\sup(a, a'))$, $B = Z_V$, and $B \to OFl(1, \ldots, \delta, 2n+1)$ the map giving the spaces of the northeast border of $OBS(Q_\circ)$, we have $\ell_4 \geq \ell_2$.

Let $\boxed{\ell_5}$ be the codimension of Z_F in the fibration

$$\pi_M^{-1}(Z_M) \to Z_M.$$

Then we have

$$\operatorname{codim}_T Z_F = \ell_1 + \ell_4 + \ell_5.$$

For a general point $t = (V., M., F_{\leq c}) \in Z_F$, consider the set $\{F_{c+1}\}$ where

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} isotropic
3. $V_{m(F_{c+1})} \subset F_{c+1} \subset V_{\mathbf{a}}^\perp$
4. $F_{c+1} \subset (F_c^\perp \cap M_{n-1})^\perp$.

Call this space ξ_t. The dimension of ξ_t is calculated here:

$$\dim \xi_t = \dim(V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - \dim(F_c) - 1 - 1.$$

We subtract 1 because we choose a line in the space $(V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp)/F_c$ and we subtract 1 for isotropy. Here, $V_{\mathbf{a}}$ is isotropic and so is $F_c^\perp \cap M_{n-1}$ since M_{n-1} is isotropic, so $V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp$ is not isotropic. Continuing the calculation:

$$
\begin{aligned}
\dim \xi_t &= \dim(V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - c - 2 \\
&= \dim((V_{\mathbf{a}} + (F_c^\perp \cap M_{n-1}))^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_{\mathbf{a}} + (F_c^\perp \cap M_{n-1})) - c - 2 \\
&= 2n + 1 - \dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_{\mathbf{a}} \cap F_c^\perp \cap M_{n-1}) - c - 2 \\
&= 2n + 1 - \dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_{\mathbf{a}} \cap M_{n-1}) - c - 2.
\end{aligned}
$$

Note that for the last step of the calculation we have $V_{\mathbf{a}} \subset V_{m(F_{\underline{c}})} \subset F_{\underline{c}} = F_c^\perp$.

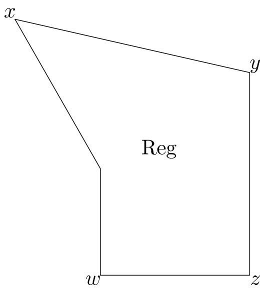


Fig. 11. A general outline of the region defined by **xyzw**.

Recall from Definition 13 that $\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \operatorname{codim}_Q W_\circ + \operatorname{codim}_Q W_{\bullet\bullet_{next}}$. So we need

$$\begin{aligned}\dim(Q_t) &= \dim(V_{\inf}^\perp) - c - 2\\ &= 2n+1-\dim(V_{\inf}) - c - 2\end{aligned}$$

$$\begin{aligned}\dim(W_\circ)_t &= \dim(V_{\mathbf{a}}^\perp \cap V_{\inf}^\perp) - c - 2\\ &= 2n+1-\dim(V_{\mathbf{a}}) - c - 2\end{aligned}$$

$$\begin{aligned}\dim(W_{\bullet\bullet_{next}})_t &= \dim(V_{\inf}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - c - 2\\ &= \dim((V_{\inf} + (F_c^\perp \cap M_{n-1}))^\perp) - c - 2\\ &= 2n+1-\dim(V_{\inf} + (F_c^\perp \cap M_{n-1})) - c - 2\\ &= 2n+1-\dim(V_{\inf}) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_{\inf} \cap F_c^\perp \cap M_{n-1}) - c - 2\\ &= 2n+1-\dim(F_c^\perp \cap M_{n-1}) - c - 2\end{aligned}$$

and

$$\begin{aligned}\operatorname{codim}_Q(W_\circ) &= \dim(Q_t) - \dim(W_\circ)_t\\ &= \dim(V_{\mathbf{a}}) - \dim(V_{\inf})\end{aligned}$$

$$\operatorname{codim}_Q(W_{\bullet\bullet_{next}}) = \dim(F_c^\perp \cap M_{n-1}) - \dim(V_{\inf}).$$

So

$$\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \dim(V_{\mathbf{a}}) + \dim(F_c^\perp \cap M_{n-1}) - 2\dim(V_{\inf}). \tag{6}$$

We now calculate $\operatorname{codim}_{Q_t} \xi_t$.

$$\begin{aligned}\operatorname{codim}_{Q_t}\xi_t &= \dim Q_t - \dim \xi_t\\ &= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim Q_t - \dim\xi_t - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})\\ &= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) + 2n+1-\dim(V_{\inf}) - c - 2\\ &\quad - (2n+1-\dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_{\mathbf{a}} \cap M_{n-1}) - c - 2)\\ &\quad - (\dim(V_{\mathbf{a}}) + \dim(F_c^\perp \cap M_{n-1}) - 2\dim(V_{\inf}))\\ &= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - [\dim(V_{\mathbf{a}} \cap M_{n-1}) - \dim(V_{\inf})]\\ &= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - \ell_3\end{aligned}$$

where $\boxed{\ell_3} = \dim(V_{\mathbf{a}} \cap M_{n-1}) - \dim(V_{\inf})$. Let $\boxed{\ell_6}$ be the codimension of the fiber $p_F^{-1}(t) \subset Z \to Z_F$ in ξ_t. See Diagram (5). Then $\operatorname{codim}_{Q_t}(p_F^{-1}(t)) = \operatorname{codim}_{Q_t}\xi_t + \ell_6$. And we have

$$\begin{aligned}\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) &= \ell_1 + \ell_4 + \ell_5 + \operatorname{codim}_{Q_t}\xi_t + \ell_6 - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})\\ &= \ell_1 + \ell_4 + \ell_5 + \ell_6 - \ell_3.\end{aligned}$$

Now, ℓ_5 and ℓ_6 are codimensions, so $\ell_5, \ell_6 \geq 0$. And $\ell_4 \geq \ell_2$, so

$$\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3.$$

Since Z is a component of $W_\circ \cap W_{\bullet\bullet_{next}}$, it must be true that $\operatorname{codim} Z \leq \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$. So $0 \geq \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3$.

We now show that $\ell_1 + \ell_2 - \ell_3 \geq 0$. Label vertex $\mathbf{m}$ of $Q_\circ$ with the value $\dim(V_m \cap M_{n-1})$ for a general point of Z. So inf is labeled $\dim(V_{\inf})$ and $\mathbf{a}''$ is labeled $\dim(V_{\mathbf{a}''})$. Consider the region defined by $\mathbf{x} = \inf$, $\mathbf{y} = \mathbf{a}''$, $\mathbf{z} = \sup$, and $\mathbf{w} = \mathbf{a}$.

Definition 14. Name the northwest, northeast, southeast, and southwest vertices $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w}$ respectively. See Fig. 11. The region defined by **xyzw** has boundary edges defined as follows: For the northern boundary (between $\mathbf{x}$ and $\mathbf{y}$), choose the southern most path from $\mathbf{y}$ to $\mathbf{x}$ such that if $\mathbf{m} \in Q_\circ$ is a node on the path, then $\mathbf{x} \prec \mathbf{m} \prec \mathbf{y}$. For the eastern boundary (between $\mathbf{z}$ and $\mathbf{y}$), choose the western most path from $\mathbf{y}$ to $\mathbf{z}$ such that if $\mathbf{m} \in Q_\circ$ is a node on the path, then $\mathbf{y} \prec \mathbf{m} \prec \mathbf{z}$.

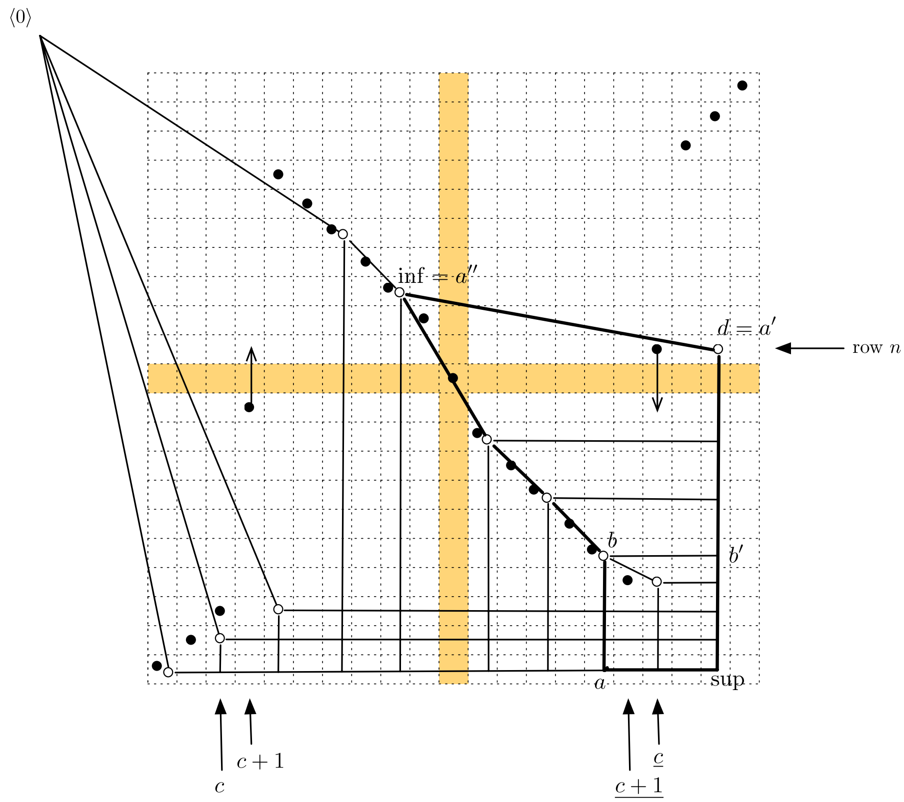


Fig. 12. Example of an s_0 move where $\inf = a''$.

Similarly, the southern boundary is the northern most path between **z** and **w** and the western boundary is the eastern most path between **x** and **w**.

Definition 15. We label each element **m** of the quilt with label $(\mathbf{m}) = \dim(V_m \cap M)$. For each quadrilateral in $Q_\circ$, define the *content* of the quadrilateral as

$$\text{label}\,(\mathbf{m}_{NE}) + \text{label}\,(\mathbf{m}_{SW}) - \text{label}\,(\mathbf{m}_{NW}) - \text{label}\,(\mathbf{m}_{SE}) \tag{7}$$

where $\mathbf{m}_{NE}$ is the northeast element of the quadrilateral, $\mathbf{m}_{SW}$ is the southwest element of the quadrilateral, $\mathbf{m}_{NW}$ is the northwest element of the quadrilateral, and $\mathbf{m}_{SE}$ is the southeast element of the quadrilateral.

The *total content* of the quadrilaterals in this region is the sum over the content of all quadrilaterals in the region. This is a linear combination of the labels of the vertices.

The net contribution of a vertex $\mathbf{m} \in Q_\circ$ is the number of quadrilaterals in the region of which it is the northeast or southwest corner, minus the number of which it is the northwest or southeast corner. Hence the only non-zero contribution to the total content is the following (see also [14]):

- Any internal diagonal edge contributes the label of its larger edge minus the label of its smaller edge (a non-negative contribution).
- The northeast and southwest corner vertices contribute their labels and the northwest and southeast corner vertices contribute the negative of their labels.

The total content, TC, of region $\inf \mathbf{a}'' \sup \mathbf{a}$ is:

$$\begin{aligned} TC &= (\text{internal diagonal contribution}) + \dim(V_{\mathbf{a}} \cap M_{n-1}) \\ &\quad + \dim(V_{\mathbf{a}''} \cap M_{n-1}) - \dim(V_{\inf} \cap M_{n-1}) - \dim(V_{\sup} \cap M_{n-1}) \\ &\geq -(\dim(V_{\sup} \cap M_{n-1}) - \dim(V_{\mathbf{a}''})) + (\dim(V_{\mathbf{a}} \cap M_{n-1}) - \dim(V_{\inf})) \\ &= -\ell_2 + \ell_3. \end{aligned}$$

The content is bounded above by $|S|$ which in turn is bounded above by ℓ_1, so $\ell_1 \geq |S| \geq -\ell_2 + \ell_3$ and we have $\ell_1 + \ell_2 - \ell_3 \geq 0$. This means $0 = \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \ell_1 + \ell_2 - \ell_3$ and we have equality on all inequalities. In particular, $Z_V = OBS(Q_\circ)_S$, $\ell_2 = \ell_4$, $\ell_5 = \ell_6 = 0$, the internal diagonal contribution is zero, and $\ell_1 = |S| =$ total content.

Internal diagonal contribution is zero which implies that all internal diagonals have the same labels on either end. Let **d** be the vertex of the white checker in row n, then $\inf \mathbf{d}$ is an internal diagonal and inf and **d** have the same label: dim(inf).

By Lemma 5.5(b)(ii) in [14], we can deduce that $\mathbf{a}''$ and $\mathbf{a}'$ have the same label. So we have

$$\dim(V_{\mathbf{a}'} \cap M_{n-1}) = \dim(V_{\mathbf{a}''} \cap M_{n-1}) = \dim(V_{\mathbf{a}''}) < \dim(V_{\mathbf{a}'}).$$

So $V_{\mathbf{a}'} \not\subset M_{n-1}$. By Lemma 2, since we have the equality $\ell_4 = \ell_2$, it must be that either $\ell_2 = 0$ or $V_{j+1} \subset M_R$, i.e. $V_{\mathbf{a}'} \subset M_{n-1}$, for all points in Z. Since this is not the case, $\ell_2 = 0$.

By Lemma 5.5(b)(i) in [14], we work our way south from the internal diagonal inf $\mathbf{d}$ to conclude that inf and $\mathbf{a}$ have the same label. Namely, $\dim(V_{\inf}) = \dim(V_{\mathbf{a}} \cap M_{n-1})$, so $\ell_3 = 0$. Which also gives us $\ell_1 = 0$.

Thus $\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \ell_1 + \ell_2 - \ell_3 = 0$. So for the s_0-case: no white checker in column $c+1$, white checker in row n, inf $\neq \mathbf{a}, \mathbf{a}''$, we have described an open subscheme of Z explicitly as a tower of projective and quadric bundles over $OBS(Q_\circ)_{\emptyset = S}$. Thus Z is unique and we've shown Z has the expected codimension.

Case (**iii**) inf $= \mathbf{a}''$. This case occurs if and only if there are no white checkers above row $R+1 = n$ in columns $\underline{c}$ or greater. See Fig. 12 as an example. The argument for the case inf $\neq a, a''$ applies verbatim until we conclude that $0 \geq \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3$. We now show $\ell_1 + \ell_2 - \ell_3 \geq 0$.

Consider the region defined by vertices a'', a', sup, and a. Label vertex $\mathbf{m} \in Q_\circ$ with $\dim(V_m \cap M_{n-1})$. Now, $V_{\mathbf{a}''}$ is a hyperplane in $V_{\mathbf{a}'}$ so

$$\dim(V_{\mathbf{a}'} \cap M_{n-1}) - \dim(V_{\mathbf{a}''} \cap M_{n-1}) = \epsilon$$

where $\epsilon = 0$ or 1. The total content, TC, of region $a''a'\sup a$ is

$$\begin{aligned} TC &= (\text{internal diagonal contribution}) + \dim(V_{\mathbf{a}} \cap M_{n-1}) + \dim(V_{\mathbf{a}'} \cap M_{n-1}) \\ &\quad - \dim(V_{\mathbf{a}''} \cap M_{n-1}) - \dim(V_{\sup} \cap M_{n-1}) \\ &= (\text{internal diagonal contribution}) + \dim(V_{\mathbf{a}} \cap M_{n-1}) + (\epsilon + \dim(V_{\mathbf{a}''} \cap M_{n-1})) \\ &\quad - \dim(V_{\mathbf{a}''} \cap M_{n-1}) - \dim(V_{\sup} \cap M_{n-1}) \\ &\geq \epsilon + (\dim(V_{\mathbf{a}} \cap M_{n-1}) - \dim(V_{\mathbf{a}''} \cap M_{n-1})) - (\dim(V_{\sup} \cap M_{n-1}) - \dim(V_{\mathbf{a}''} \cap M_{n-1})) \\ &= \epsilon + \ell_3 - \ell_2. \end{aligned}$$

TC is bounded above by ℓ_1 so $\ell_1 \geq \epsilon + \ell_3 - \ell_2$ and thus $\ell_1 + \ell_2 - \ell_3 \geq \epsilon$. So we have $0 \geq \ell_1 + \ell_2 - \ell_3 \geq \epsilon$ resulting in $\epsilon = 0$ and equality holding in all previous inequalities. In particular, $\ell_4 = \ell_2$ and the the internal diagonal contribution is zero. Using Lemma 5.5(b)(i) from [14] and moving south from edge $a''a'$, we have $\ell_3 = 0$.

Since $\epsilon = 0$, we have

$$\dim(V_{\mathbf{a}'} \cap M_{n-1}) = \dim(V_{\mathbf{a}''} \cap M_{n-1}) = \dim(V_{\mathbf{a}''}) < \dim(V_{\mathbf{a}'})$$

so $V_{\mathbf{a}'} \not\subset M_{n-1}$. By Lemma 2, with the equality $\ell_2 = \ell_4$ and $V_{\mathbf{a}'} \not\subset M_{n-1}$, we must have $\ell_2 = 0$. Thus $\ell_4 = \ell_2 = 0$ and we have $0 = \ell_1 + \ell_2 - \ell_3 = \ell_1$. So for inf $= a''$ we have described an appropriate subscheme of $Z$and thus completed the proof in this case. □

Irreducible components of D_Q are a subset of the D_S's. We describe the components of D_Q in terms of strata of $OBS(Q_\circ)$. Let $\mathbf{d}$ be the vertex of $Q_\circ$ where the white checker in row n is located. Define the *western good quadrilaterals* of $Q_\circ$ to be those quads with eastern two vertices dominating $\mathbf{d}$ and western two vertices dominated by $\mathbf{a} = m(\underline{F_{c+1}})$. Let the *eastern good quadrilaterals* be those quads whose vertices all dominate $\mathbf{d}$, and are east of a western good quad. Let $\mathbf{b}$ be the southwestern corner of the region of good quads and $\mathbf{b}'$ be the southeastern corner of the region of good quads. The region of good quads is $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}'\mathbf{b}'\mathbf{b}$ (possibly empty). It may be helpful to refer to Fig. 10.

Define $W_{\bullet_{next}} \subset Q$, fibered over T, with fibers $\{F_{c+1}\}$ such that

1. $(t, F_{c+1}) \in Q_t$
2. $(F_c^\perp \cap M_n) \subset F_{c+1}^\perp$
3. $(F_c^\perp \cap M_{n+1}) \not\subset F_{c+1}^\perp$ (this is an open condition).

In other words, $W_{\bullet_{next}}$ is the pullback of the Cartier divisor $X_{\bullet_{next}} \subset X_{\bullet\bullet_{next}}$ to $W_{\bullet\bullet_{next}}$. Let D_Q be the pullback of the Cartier divisor $X_{\bullet_{next}} \subset X_{\bullet\bullet_{next}}$ to the irreducible variety $W_\circ \cap W_{\bullet\bullet_{next}} \subset Q$. Thus $D_Q = W_{\bullet_{next}} \cap W_\circ \subset W_{\bullet\bullet_{next}} \cap W_\circ$.

Let S be a set of good quadrilaterals with none weakly southeast of another. Define a subvariety D_S of $W_{\bullet_{next}} \cap W_\circ$ as follows. Let T_S be the open subvariety of the the pullback of $OBS(Q_\circ)_S$ to T, on which $\dim(V_{\mathbf{a}} \cap M_n)$ is constant. Let D_S be the closure in D_Q of the pullback of T_S to $D_Q \subset W_\circ \cap W_{\bullet\bullet_{next}}$. T is irreducible and fibers over general points of $T_S \subset T$ are irreducible and equidimensional (equidimensional because $\dim(V_{\mathbf{a}} \cap M_n)$ is constant on T_S). So the pullback of T_S to D_Q is irreducible, implying that D_S is irreducible.

Let S run over all subsets of good quads with none weakly southeast of another. Let Z be an irreducible component of D_Q (not to be confused with Z used in earlier proofs). We will show that there is a set S such that $Z = D_S$.

Theorem 4.4. *The irreducible components of D_Q are a subset of the set of D_S where S is some set of good quadrilaterals with none weakly southeast of another.*

Proof. Here we will use n and $\mathbf{a}'$ instead of $n-1$ and $\mathbf{a}''$, in particular, $\inf = \inf(a, a')$.
Case (**i**) $\inf(\mathbf{a}, \mathbf{a}') = \mathbf{a}$.

Then $W_{\bullet_{next}} \cap W_\circ = W_{\bullet_{next}}$ and $W_{\bullet_{next}} = D_\emptyset$ since there are no good quadrilaterals.

Case (**ii**) $\inf(\mathbf{a},\mathbf{a}') \neq \mathbf{a}$

Z is an irreducible component of D_Q, so Z has the same dimension as D_Q and D_Q is a divisor of $W_{\bullet\bullet_{next}} \cap W_\circ$ so

$$\operatorname{codim}_Q Z = \operatorname{codim}_Q (W_{\bullet\bullet_{next}} \cap W_\circ) + 1.$$

Let $Z_{OBS(Q_\circ)}, Z_M, Z_F$ be the image of Z in $OBS(Q_\circ)$, $OBS(Q_\circ) \times \{M.\}$, and $T \subset OBS(Q_\circ) \times \{M.\} \times \{F_{\leq c}\}$ respectively. Let $\boxed{\ell_1} = \operatorname{codim}_{OBS(Q_\circ)} Z_{OBS(Q_\circ)}$ and let S be the set of (at most ℓ_1) quadrilaterals corresponding to the smallest closed stratum of $OBS(Q_\circ)$ in which $Z_{OBS(Q_\circ)}$ is contained. Let $\boxed{\ell_4}$ be the codimension of Z_M in $\pi_M^{-1}(Z_M)$ and for a general point in Z, let

$$\boxed{\ell_2} = \dim(V_{\sup(\mathbf{a},\mathbf{a}')} \cap M_n) - \dim(V_{\mathbf{a}'}).$$

Using Lemma 2, let $R = n$, $j = \dim(V_{\mathbf{a}'})$, and $B = Z_{OBS(Q_\circ)}$. Then $\ell_4 \geq \ell_2$. Note: $R = a_j$ since there is a white checker in row $R = n$ and so $a_j = n = R$. Thus we get equality ($\ell_4 = \ell_2$) only if $\ell_2 = 0$.

Let $\boxed{\ell_5}$ be the codimension of Z_F in $\pi_F^{-1}(Z_F)$. So we get as before,

$$\operatorname{codim}_T Z_F = \ell_1 + \ell_4 + \ell_5.$$

For a general point $t = (V., M., F_{\leq c}) \in Z_F$, consider the set $\{F_{c+1}\}$ where

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{m(F_{c+1})} \subset F_{c+1} \subset V^\perp_{m(\underline{F_{c+1}})}$
4. $F_{c+1} \subset (F_c^\perp \cap M_n)^\perp$.

We will call this space ξ_t. For general $t \in Z_F$, the dimension of ξ_t is

$$\begin{aligned}
\dim \xi_t &= \dim(F_c^\perp \cap V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_n)^\perp) - \dim(F_c) - 1 - 1\\
&= \dim(V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_n)^\perp) - c - 2\\
&= \dim((V_{\mathbf{a}} + (F_c^\perp \cap M_n))^\perp) - c - 2\\
&= 2n + 1 - \dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_n) + \dim(V_{\mathbf{a}} \cap F_c^\perp \cap M_n) - c - 2\\
&= 2n + 1 - \dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_n) + \dim(V_{\mathbf{a}} \cap M_n) - c - 2.
\end{aligned}$$

In line one of the above calculation, the first -1 is for choosing a line in the space given. The second -1 is for the condition that the line must be isotropic. Here, $V_{\mathbf{a}}^\perp \cap (F_c^\perp \cap M_n)^\perp$ is not isotropic so this is a nontrivial condition.

We now calculate the codimension of ξ_t in Q_t. Note that $\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \operatorname{codim}_Q (W_\circ \cap W_{\bullet\bullet_{next}})$ was shown in the previous proof.

$$\begin{aligned}
\operatorname{codim}_{Q_t} \xi_t &= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim Q_t - \dim \xi_t - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})\\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + (2n + 1 - \dim(V_{\inf(a,a'')}) - c - 2)\\
&\quad - (2n + 1 - \dim(V_{\mathbf{a}}) - \dim(F_c^\perp \cap M_n) + \dim(V_{\mathbf{a}} \cap M_n) - c - 2)\\
&\quad - (\dim(V_{\mathbf{a}}) + \dim(F_c^\perp \cap M_{n-1}) - 2\dim(V_{\inf}))\\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim \inf + \dim(F_c^\perp \cap M_n)\\
&\quad - \dim(F_c^\perp \cap M_{n-1}) - \dim(V_{\mathbf{a}} \cap M_n)\\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim \inf + 1 - \dim(V_{\mathbf{a}} \cap M_n).
\end{aligned}$$

In the last line of the calculation above, $\dim(F_c^\perp \cap M_n) = \dim(F_c^\perp \cap M_{n-1}) + 1$.

Let

$$\boxed{\ell_3} = \dim(V_{\mathbf{a}} \cap M_n) - \dim(V_{\inf}).$$

Then $\ell_3 \geq 0$ because $V_{\inf} \subset V_{\mathbf{a}} \cap M_n$. This step contributes to a codimension of $\boxed{1 - \ell_3}$ compared to $\operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})$.

Let $\boxed{\ell_6}$ be the codimension of the fiber $p_F^{-1}(t) \subset Z \to Z_F$ in ξ_t. Picture a diagram similar to Diagram (5). Then $\operatorname{codim}_{Q_t}(p_F^{-1}(t)) = \operatorname{codim}_{Q_t} \xi_t + \ell_6$. So we have

$$\begin{aligned}
1 &= \operatorname{codim} Z - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})\\
&= (\ell_1 + \ell_4 + \ell_5 + \operatorname{codim} \xi_t + \ell_6) - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})\\
&= \ell_1 + \ell_4 + \ell_5 + (\operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + 1 - \ell_3) + \ell_6 - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})\\
&= 1 + \ell_1 + \ell_4 + \ell_5 - \ell_3 + \ell_6\\
&\geq 1 + \ell_1 + \ell_2 - \ell_3.
\end{aligned}$$

The final inequality is because $\ell_4 \geq \ell_2$ and $\ell_5, \ell_6 \geq 0$. We will now show that $\ell_1 + \ell_2 - \ell_3 \geq 0$.

Label vertex $\mathbf{m}$ of $Q_\circ$ with $\dim(V_m \cap M_n)$. We will compute the content of the region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. Each internal diagonal edge contributes the label of its larger vertex minus the label of its smaller vertex, a non-negative contribution. The defining corners contribute their labels (positive for $\mathbf{a}$, $\mathbf{a}'$ and negative for $\inf(\mathbf{a}, \mathbf{a}')$ and $\sup(\mathbf{a}, \mathbf{a}')$). Thus the total content is

$$\begin{aligned} TC &= (\text{internal diagonal contribution}) + \dim(V_{\mathbf{a}} \cap M_n) + \dim(V_{\mathbf{a}'} \cap M_n) \\ &\quad - \dim(V_{\inf} \cap M_n) - \dim(V_{\sup} \cap M_n) \\ &\geq (\dim(V_{\mathbf{a}'}) - \dim(V_{\sup} \cap M_n)) + (\dim(V_{\mathbf{a}} \cap M_n) - \dim(V_{\inf})) \\ &= -\ell_2 + \ell_3. \end{aligned}$$

Total content is bounded above by $|S|$ which is bounded above by ℓ_1. This implies $\ell_1 \geq |S| \geq -\ell_2+\ell_3$ and so $\ell_1+\ell_2-\ell_3 \geq 0$, which gives us

$$1 \geq 1 + \ell_1 + \ell_2 - \ell_3 \geq 1 + 0 = 1.$$

So $\ell_1 + \ell_2 - \ell_3 = 0$. Thus equality holds in all inequalities above. In particular, $\ell_5 = \ell_6 = 0$ and $\ell_2 = \ell_4 = 0$ and $\ell_1 = \ell_3$. Note that ℓ_1 and ℓ_3 are not necessarily zero. And $Z_{OBS(Q_\circ)}$ is the stratum corresponding to S. And so all quadrilaterals have content zero except for ℓ_1 quads with content 1 in region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. We will consider two cases here:

Case $\mathbf{b} \neq \mathbf{a}$

The reader may wish to refer to Fig. 10 for an example. Let $\mathbf{b}'' \in Q_\circ$ be the vertex of the other end of the northern most diagonal edge emanating southeast from $\mathbf{b}$. By the equality above, the internal diagonal contribution is zero, so $\mathbf{b}$ and $\mathbf{b}''$ have the same label. Applying Lemma 5.5(b)(i) from [14] to the region below edge $\mathbf{bb}''$, we have that all vertices below $\mathbf{bb}''$ have the same label as well. In particular, the labels of $\mathbf{b}$ and $\mathbf{a}$ are the same. Let E be the set of edges due south of $\mathbf{b}''$ union the edge $\mathbf{bb}''$. By midsort Conjecture 2, there are no white checkers in the region directly east of E, so this region is a grid of quadrilaterals. Using Lemma 5.5(b)(ii) from [14], we get that the labels on $\mathbf{b}'$ and $\sup(\mathbf{a}, \mathbf{a}')$ are the same. Thus, we do not add content by this new region east of E, and so the total content of the region of good quads, $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}'\mathbf{b}'\mathbf{b}$, is the same as the content of the region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. This content is ℓ_1. Thus the ℓ_1 positive-content quadrilaterals S are a subset of the good quads.

Case $\mathbf{b} = \mathbf{a}$

The result that the ℓ_1 positive-content quads S are a subset of the good quads is immediate.

We now show that no element of S is weakly southeast of another. This portion of the proof is exactly Section 5.11 in [14]. We include the paragraph here for completeness.

Fix a positive-content quadrilateral. Then its northeast, southeast, and southwest vertices have the same label. Thus by repeated application of Lemma 5.5(b)(i) from [14], all vertices south of its southern edge are labeled the same, and there are no positive-content quadrilaterals (elements of S) south of this edge. Let E' be the union of edges due south of the northeast vertex of our positive-content quadrilateral. Repeated applications of Lemma 5.5(b)(ii) from [14] imply that any two vertices east of E' in the same column have the same label, and there are no positive content quadrilaterals here either.

Thus $Z = D_S$ for the S described above and we've shown that the irreducible components of D_Q are a subset of $\{D_S\}_S$. □

Contraction of all but one or two divisors by π. We show in this section that all divisors but possibly $D_\emptyset$ and $D_{\text{NW good quad}}$ are contracted by π. Part (a) of the theorem shows that all other D_S are contracted by π. Part (b) shows that $D_\emptyset$ is contracted by π when predicted. The strategy is as follows: we will construct for a general point $(V, M_., F_.) \in \pi(D_S)$ a positive dimensional family in D_S which collapses to $(V, M_., F_.)$. This will prove that D_S is contracted by π to a component of codimension greater than one in $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}X_{\circ\bullet}$, hence does not contribute to D.

Theorem 4.5. (a) *If $S \neq \emptyset$ and $S \neq \{$northwest good quad$\}$ then D_S is contracted by π.*

(b) *If $S = \emptyset$ and the white checker in row n is in the descending checker's square $(n, \underline{c})$ and there is a white checker in a column $n + 2 \leq col \leq \underline{c+1}$ then D_S is contracted by π.*

Proof of part (a). Given a general point of D_S, $((V_m)_{\mathbf{m}\in Q_\circ}, M_., F_.) \in D_S$, we will produce a one-parameter family $((V'_m)_{\mathbf{m}\in Q_\circ}, M_., F_.)$ through $(V_m)_{\mathbf{m}\in Q_\circ}$ in the stratum $OBS(Q_\circ)_S$, fixing those V_m on the northeast border of $OBS(Q_\circ)$ and those V_m where $\mathbf{a} \prec \mathbf{m}$ along the southwest border and any $\mathbf{m}$ along the southwest border in checker board columns $1, \ldots, c$. Note: For $1 \leq i \leq 2n + 1$ we have $V_{m(M_i)} \subset M_i$ since $V_{m(M_i)}$ is on the northeast border so is fixed. Also note that for $1 \leq j \leq c + 1$ and $\underline{c+1} \leq j \leq 2n + 1$ we have $V_{m(F_j)} \subset F_j \subset V^\perp_{m(F_j)}$. These two comments hold for any element $((V'_m)_{\mathbf{m}\in Q_\circ}, M_., F_.)$ in the family we will describe. Note also that $V_{max(Q_\circ)}$ is fixed so this one-parameter family in D_S will be contracted by π.

Here is a description of the one-parameter family: The description is exactly as in the proof of Proposition 5.13(a) in [14]. We reiterate the proof here for the purpose of checking details.

Choose a quadrilateral **stuv** in S. Name the elements of $Q_\circ$ as in Fig. 13. $\mathbf{g_m}$ is the white checker in the column containing s. $\mathbf{f_{m-1}}$ is the next white checker to the west of $\mathbf{g_m}$. A few comments:

1. $\mathbf{g_m}$ is not necessarily a vertex within the "good quad" region; it may be north of the region.
2. The $\mathbf{s}, \mathbf{g_1}, \ldots, \mathbf{g_m}$ column is never a subset of the northeast border because there is always a column east of $\mathbf{s}, \mathbf{g_1}, \ldots, \mathbf{g_m}$ and the white checker in that column is in a more northern row than $\mathbf{g_m}$ by midsort Conjecture 2.

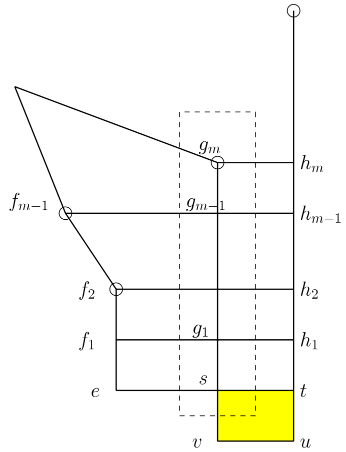


Fig. 13. Let quadrilateral $stuv$ be an element of S. Label the elements of $Q_\circ$ as in this figure.

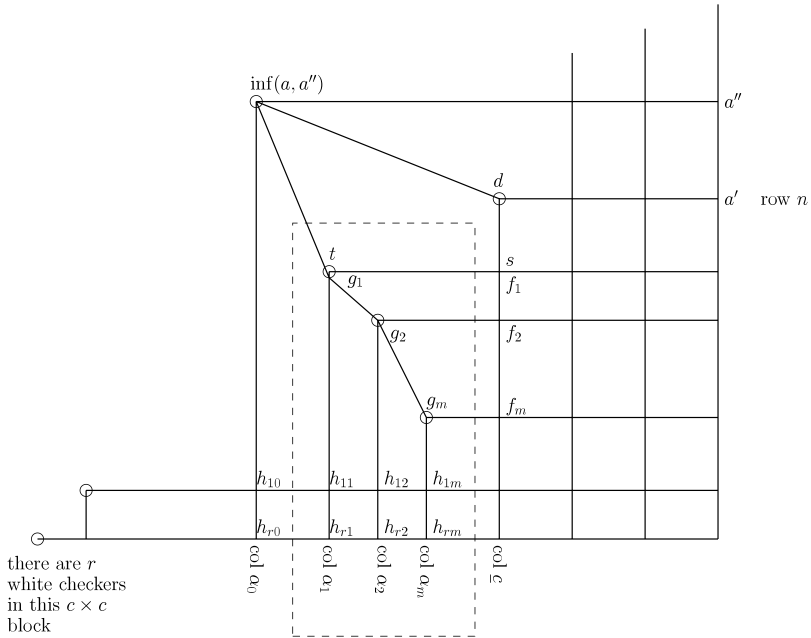


Fig. 14. Diagram for $S = \emptyset$ and there is a white checker in row n in the descending black checker's square.

3. If **stuv** is the northwest good quad then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$. So if $\inf(\mathbf{a}, \mathbf{a}'') = \mathbf{a}''$ then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$ is on the northeast border and is required to be fixed. And if $\inf(\mathbf{a}, \mathbf{a}'') \neq \mathbf{a}, \mathbf{a}''$ then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$ has a third southeastern edge pointing due east (toward $\mathbf{a}''$).

So when **stuv** is *not* the northwest good quad, then $\mathbf{g_m}$ always has exactly one edge pointing northwest and two edges pointing southeast.

We define our family as follows: Let $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{s}, \mathbf{g_1}, \ldots, \mathbf{g_m}$. Then choose V'_s from the open set of $\mathbb{P}(V_v/V_e) \cong \mathbb{P}^1$ such that $\dim(V'_{g_i}) = \dim(\mathbf{g_i})$ for $1 \leq i \leq m$ and V'_{g_i} is defined as $V'_{g_i} = V'_s \cap V_{h_i}$. We do not get the full $\mathbb{P}^1$ of choices here because we must choose V'_s so its intersection with the "$\mathbf{h_i}$ column" gives spaces with the expected codimensions. Note that $V'_{f_i} = V_{f_i}$ is contained in V'_{g_i} since $V'_{g_i} = V'_s \cap V_{h_i}$, $V_{f_i} = V'_{f_i} = V_e \cap V_{h_i}$, and $V_e \subset V'_s$ so our containments $V'_{f_i} \subset V'_{g_i}$ make sense. $\square$

Proof of part (b). We now suppose that $S = \emptyset$ and there is a white checker in the descending black checker position, $(n, \underline{c})$. Consider Fig. 14. Call the white checker in position $(n, \underline{c})$, $\mathbf{d}$. Let $\mathbf{t}$ be the northwestern-most white checker in columns $n + 2 \leq col \leq \underline{c+1}$. $\mathbf{g_1}, \ldots, \mathbf{g_m}$ are the m white checkers in columns $n + 2 \leq col \leq \underline{c+1}$. These white checkers are in columns $n + 2 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m \leq \underline{c+1}$. Let r_{α_i} be the row of the white checker called g_i. By midsort Conjecture 3, $n + 2 < r_{\alpha_1} < r_{\alpha_2} < \cdots < r_{\alpha_m} \leq 2n + 1$. $\mathbf{h_{ij}} \in Q_\circ$ is in column α_j and the row of the ith white checker in the southwest $c \times c$ block. There are r such white checkers in this block.

In $\bullet_{next}$, the black checker configuration tells us that

$$F_{\underline{c}} \cap M_n = F_{\underline{c+1}} \cap M_n = F_{\underline{c+2}} \cap M_n = \cdots = F_{n+2} \cap M_n.$$

In particular,

$$F_{\underline{c}} \cap M_n = F_{\alpha_m} \cap M_n = \cdots = F_{\alpha_1} \cap M_n$$

where a_i is the column of the ith white checker in the region weakly south of the critical diagonal.

We are given $((V_m), M_., F_.)$ such that

1. $(M_., F_.) \in X_{\bullet_{next}}$
2. $(V_m)_{m \in Q_\circ} \in OBS(Q_\circ)_\emptyset$
3. $V_m \subset M_{\text{row of } m} \cap F_{\text{col of } m}$ for all $\mathbf{m} \in Q_\circ$.

Some other observations:

1. $S = \emptyset$ so we may assume that $V_m \neq V_{m'}$ for $\mathbf{m}, \mathbf{m}'$ opposite corners of any quadrilateral of $Q_\circ$. In particular, $V_d \neq V_t$
2. $V_d \subset M_n \cap F_{\underline{c}} = M_n \cap F_{\alpha_1}$
3. $V_t \subset F_{\alpha_1}$
4. $V_s = \langle V_d, V_t \rangle$ so $V_s \subset F_{\alpha_1}$. Note also that $V_s = \langle V_d, V_{g_1} \rangle$ and $V_{f_j} = \langle V_d, V_{g_j} \rangle$ with $V_{g_j} \subset F_{\alpha_j}$ and $V_d \subset M_n \cap F_{\underline{c}} = M_n \cap F_{\alpha_j}$ so $V_{f_j} \subset F_{\alpha_j}$ for $1 \leq j \leq m$.

We will describe an m-dimensional family $(V'_w)_{\mathbf{w} \in Q_\circ}$ through $(V_w)_{\mathbf{w} \in Q_\circ}$ in the stratum $OBS(Q_\circ)_\emptyset$ preserving all spaces on the northeast border and all spaces on the southwest border in columns $1 \leq col < n + 1$ and in columns $\underline{c} \leq col \leq 2n + 1$. In particular, we fix V_{max}.

Let $V'_w = V_w$ for $\mathbf{w} \neq \mathbf{g_1}, \ldots, \mathbf{g_m}$ and $\mathbf{w} \neq \mathbf{h_{ij}}$ for $1 \leq i \leq r$ and $1 \leq j \leq m$. Choose V'_{g_1} from the open subset of $\mathbb{P}(V_s/V_{\inf(a,a')})$ such that $V'_{g_1} \neq V_d$ and $V'_{h_{i1}} = \langle V_{h_{i0}}, V'_{g_1} \rangle$ has dim h_{i1} for $1 \leq i \leq r$. Some notes: Since $V_s \subset F_{\alpha_1}$, we know that $V'_{g_1} \subset F_{\alpha_1}$ also. And V'_{g_1} has the correct $M_.$ row containment because V_s is in the same row. And $V'_{h_{i1}} = \langle V_{h_{i0}}, V'_{g_1} \rangle$ has the correct $M_.$ and $F_.$ containment because of $V_{h_{i0}}$ and V'_{g_1} have the correct containments. In particular, $V'_{h_{r1}} \subset F_{\alpha_1}$ because $V'_{g_1} \subset F_{\alpha_1}$ and $V_{h_{r0}} \subset F_{n+1} \subset F_{\alpha_1}$.

Now, for $2 \leq j \leq m$, choose V'_{g_j} from the open subset of $\mathbb{P}(V_{f_j}/V'_{g_{j-1}})$ such that $V'_{g_j} \neq V_{f_{j-1}}$ and $V'_{h_{ij}} = \langle V'_{g_j}, V'_{h_{i,j-1}} \rangle$ has dim h_{ij} for $1 \leq i \leq r$. Note that $V'_{g_j} \subset V_{f_j}$ and $V_{f_j} \subset F_{\alpha_j}$ so $V'_{g_j} \subset F_{\alpha_j}$. And $V'_{h_{rj}} = \langle V'_{h_{r,j-1}}, V'_{g_j} \rangle \subset F_{\alpha_j}$ because $V'_{h_{r,j-1}} \subset F_{\alpha_{j-1}} \subset F_{\alpha_j}$ and $V'_{g_j} \subset F_{\alpha_j}$.

Since the original point $((V_w)_{w \in Q_\circ}, M_., F_.)$ is in this family, it's nonempty. So we've described an m-dimensional ($m \geq 1$) family in D_S that collapses when we apply π. Thus D_S is contracted by π. $\square$

Multiplicity 1. We now show that the D_S that are not contracted by π appear with multiplicity 1 in the Cartier divisor D_Q.

Theorem 4.6. (a) *When $D_\emptyset$ is not contracted by π, the multiplicity of the Cartier divisor D_Q along the Weil divisor $D_\emptyset$ is* 1.
(b) *If there are good quadrilaterals, the multiplicity of D_Q along $D_{\{NW\,good\,quad\}}$ is* 1.

Proof of part (a). Consider a general point $(V_., M_., F_.) \in Cl_{OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$. The reader may wish to refer to Figs. 9 and 10 as examples. Without loss of generality, let $F_.$ be the standard flag where $F_j = \langle e_1, \ldots, e_j \rangle$. and $M_i = \langle e_{c_1}, \ldots, e_{c_i} \rangle$ where c_k is the column of the black checker in row k of the $\bullet$-configuration. Since we are considering a general point of $Cl_{OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$, $V \cap M_i \cap F_j = V_{\mathbf{m}}$ for $\mathbf{m} \in Q_\circ$ in position (i, j) on the checker board. In particular, $\mathbf{d}$ is the white checker in position (n, c_d) and

$$V_d = \langle V_{\inf}, w + e_{\underline{c}} \rangle$$

where

$$w = \sum_{i=1+\text{ col of inf}}^{n} a_i e_i + \sum_{i=\underline{c}+1}^{c_d} b_i e_i.$$

By $\mathbf{d}$'s position, (n, c_d), we can assume that the coefficient in front of $e_{\underline{c}}$ is 1 and $b_{c_d} \neq 0$. By hypothesis, we are looking at $D_\emptyset$ and are only interested in $D_\emptyset$ when it is not contracted by π. So by Theorem 4.5, we may assume $c_d > \underline{c}$.

We give a test family $\mathcal{F}$ through the general point $(V_., M_., F_.) \in Cl_{OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ meeting D_Q along $D_\emptyset$ with multiplicity 1. The family $\mathcal{F} = \{(V'_., M'_., F'_.)\}$ is given by

- Fix $F'_. = F_.$.
- Let $M'_i = M_i$ and $(M'_i)^\perp = M_i^\perp$ for $1 \leq i \leq n - 1$.
- Define $M'_n = M_{n-1} + \langle \frac{1}{2}s^2 e_{c+1} + ste_{n+1} - t^2 e_{\underline{c}} \rangle$ for $[s, t] \in \mathbb{P}^1$. Then $M'_{n+1} = (M'_n)^\perp$.
- For $\mathbf{m} \in Q_\circ$ where $\mathbf{d} \not\prec \mathbf{m}$, let $V'_m = V_m$.

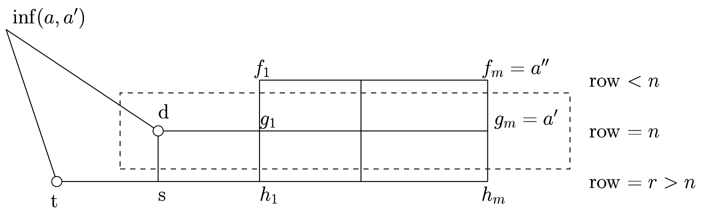


Fig. 15. Labels for $Q_\circ$ in proof of Theorem 4.6(b).

- Define

$$V'_d = V_{\inf} + \left\langle w + \frac{1}{2}s^2 e_{c+1} + ste_{n+1} - t^2 e_{\underline{c}} \right\rangle.$$

  Note that V'_d is completely determined by the choice of $[s, t]$ for M'_n.
- For $\mathbf{m} \in Q_\circ$ with $\mathbf{d} \prec \mathbf{m}$ and $\mathbf{m} \neq \mathbf{d}$, inductively define V'_m as $V'_{NE} + V'_{SW}$, the span of the vector spaces associated to the northeast and southwest corners of the quadrilateral where $\mathbf{m}$ is the southeast corner.

The set of all such $(V'_., M'_., F'_.)$ is the 1-dimensional family $\mathcal{F}$.

For a point $(V'_., M'_., F'_.) \in \mathcal{F}$, consider a quadrilateral in $Q_\circ$, with northeast corner at position (r_{NE}, c_{NE}) and associated vector space V_{NE} and with southwest corner at position (r_{SW}, c_{SW}) and associated vector space V_{SW}. V_{NE} has a vector $v = \sum_{i=1}^{2n+1} a_i e_i$ where $a_{c_{NE}} \neq 0$. If V_{NE} is in column c_d, then $a_{c_{NE}} = b_{c_d} \neq 0$ because $\langle w + (\frac{1}{2}s^2 e_{c+1} + ste_{n+1} - t^2 e_{\underline{c}})\rangle \subset V_{NE}$. If V_{NE} is in a column not equal to c_d, then V_{NE} inherits the nonzero coefficient $a_{c_{NE}}$ from the general point $(V_., M_., F_.)$. Now, because $V_{SW} \subset F_{c_{SW}} = \langle e_1, \ldots, e_{c_{SW}}\rangle$ and $c_{SW} < c_{NE}$, all vectors of V_{SW} are of the form $w = \sum_{i=1}^{2n+1} b_i e_i$ where $b_{c_{NE}} = 0$. So $V_{NE} \neq V_{SW}$ for any quad in $Q_\circ$. Thus $V'_. \in OBS(Q_\circ)_\emptyset$ for every element of $\mathcal{F}$.

When $[s, t] = [0, 1]$, $(V'_., M'_., F'_.)$ is the original general point $(V_., M_., F_.)$. So $\mathcal{F} \not\subset D_\emptyset$. When $[s, t] = [1, 0]$, $M'_n = M_{n-1} + \langle e_{c+1}\rangle$ which implies $(M'_n \cap F_{\underline{c}}^\perp) \subset F_{c+1}^\perp$, which makes $(M'_., F'_.) \in X_{\bullet_{next}}$. So $\mathcal{F}$ meets $D_\emptyset$.

Now, moving away from the original general point, let $s = 1$. Then we have $M'_n = M'_{n-1} + \langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\rangle$. A point in $\mathcal{F}$ is in $D_\emptyset$ if and only if

$$\begin{aligned} & M'_n \cap (F'_{\underline{c}})^\perp \subset (F'_{c+1})^\perp \\ \iff & \dim(M'_n \cap F_{c+1}) \geq 1 \\ \iff & \dim\left(\left(M'_{n-1} + \left\langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\right\rangle\right) \cap F_{c+1}\right) \geq 1. \end{aligned}$$

Now, for $(M'_., F'_.) \in X_\bullet \cup X_{\bullet_{next}}$, we have that $\dim(M'_{n-1} \cap F_{c+1}) = 0$, so

$$\dim\left(\left(M'_{n-1} + \left\langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\right\rangle\right) \cap F_{c+1}\right) \geq 1 \iff \left\langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\right\rangle \subset F_{c+1}.$$

This is true if and only if $-t^2 = 0$ and $t = 0$ (since $n + 1 > c + 1$ and $\underline{c} > c + 1$), a multiplicity 1 condition. So $\mathcal{F}$ meets D_Q with multiplicity 1 at a point of $D_\emptyset$, and therefore D_Q has multiplicity one along $D_\emptyset$. □

Proof of part (b). We give a test family $\mathcal{F}$ through a general point $(V_., M_., F_.)$ of $Cl_{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ meeting D_Q along $D_{\{\text{NW good quad}\}}$ with multiplicity 1. Label the elements of $Q_\circ$ as in Fig. 15. See Fig. 10 as an example.

- $\mathbf{t}$ is the highest white checker in columns $n + 2 \leq col \leq \underline{c + 1}$.
- r is the row of checker $\mathbf{t}$, $\underline{r} = 2n + 1 - r < n$.

The family $\mathcal{F} = \{(V'_., M'_., F'_.)\}$ is given by

- $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{d}, \mathbf{g_1}, \ldots, \mathbf{g_m}$
- Choose $e_t \in V_t$ and $e_d \in V_d$ so that e_t is a generator of $V_t/V_{\inf}$ and e_d is a generator of $V_d/V_{\inf}$. Let $V'_d = \langle V_{\inf}, \mu e_t + \nu e_d\rangle$ where $[\mu, \nu] \in \mathbb{P}^1$ and $V'_{g_i} = \langle V_{f_i}, V'_d\rangle$ has $\dim(V'_{g_i}) = \dim(\mathbf{g_i})$. So V'_d varies in an open subset of $\mathbb{P}(V_s/V_{\inf})$.
- For $1 \leq i \leq \underline{r}$, let $M'_i = M_i$ and $(M'_i)^\perp = M_i^\perp$.
- We make the following observation: there is no white checker in row $\underline{r} + 1$ because $\underline{r} + 1 = 2n + 1 - r + 1 = 2n + 2 - r$ and row r has checker $\mathbf{t}$ in it. By maximality, since there are no white checkers in rows $n + 2 \leq row \leq r - 1$, we must have white checkers in rows $\underline{r} + 2 \leq row \leq n$.
- Choose a line L such that

  1. $L \not\subset M_{\underline{r}}$
  2. $L \subset M_r = M_{\underline{r}}^\perp$
  3. L is isotropic

4. $L \subset ((V'_d)^{\perp} \cap V_{fm}^{\perp}) = (V_{fm}^{\perp} \cap \langle \mu e_t + \nu e_d \rangle^{\perp})$
5. $L \subset F_c^{\perp}$.

Then define

$$M'_{\underline{r}+1} = M_{\underline{r}} + L.$$

This is a valid choice for $M'_{\underline{r}+1}$ if the following are true:

1. $M_{\underline{r}} \subset M'_{\underline{r}+1} \subset M_r$
2. $\dim(M'_{\underline{r}+1}) = \underline{r} + 1$
3. $M'_{\underline{r}+1}$ is isotropic
4. $V'_{m(M_{\underline{r}+1})} \subset M'_{\underline{r}+1}$
5. $V'_{m(M_{r-1})} \subset (M'_{\underline{r}+1})^{\perp} = M'_{r-1}$.

We will show that the above are all satisfied by our choice of $M'_{\underline{r}+1}$. $M_{\underline{r}} \subset M_{\underline{r}} + L = M'_{\underline{r}+1}$ and both $M_{\underline{r}}$ and L are contained in M_r so 1 is satisfied. $L \not\subset M_{\underline{r}}$ so $\dim(M_{\underline{r}} + L) = \underline{r} + 1$, proving 2. L is isotropic and contained in $M_{\underline{r}}^{\perp}$ so $M_{\underline{r}} + L$ is isotropic which is 3. There is no white checker in row $\underline{r}$ so 4 is not a new condition. There are also no white checkers in rows $n + 1, \ldots, r - 1$ so

$$V'_{m(M_{r-1})} = V'_{m(M_n)} = V'_{gm} = \langle \mu e_t + \nu e_d \rangle + V_{fm}.$$

Now, $\langle \mu e_t + \nu e_d \rangle + V_{fm} \subset V_s \subset M_r$ which implies $M_{\underline{r}} \subset (\langle \mu e_t + \nu e_d \rangle + V_{fm})^{\perp}$. And $L \subset (\langle \mu e_t + \nu e_d \rangle + V_{fm})^{\perp}$ by hypothesis. So

$$\begin{aligned} & M_{\underline{r}} + L = M_{\underline{r}+1} \subset (\langle \mu e_t + \nu e_d \rangle + V_{fm})^{\perp} \\ \iff & \langle \mu e_t + \nu e_d \rangle + V_{fm} \subset M_{\underline{r}+1}^{\perp} \\ \iff & V'_{m(M_{r-1})} \subset M_{\underline{r}+1}^{\perp}. \end{aligned}$$

Thus showing 5.

- Now define for $\underline{r} + 2 \le i \le n$

$$M'_i = M'_{\underline{r}+1} + V'_{m(M_i)}.$$

Then with perps, we have $M'_{\cdot}$.

- We now build the $F'_{\cdot}$ part of the family. Let $F'_j = F_j$ for $1 \le j \le c$. Choose F'_{c+1} such that

1. $F_c \subset F'_{c+1} \subset F_c^{\perp}$
2. F'_{c+1} is isotropic
3. $V'_{\mathbf{a}} \subset (F'_{c+1})^{\perp}$
4. $V'_{m(F_{c+1})} \subset F'_{c+1}$ (There is no white checker in column $c + 1$, so this is not a new condition.)
5. $(M'_{n-1} \cap F_c^{\perp}) \subset (F'_{c+1})^{\perp}$.

Consider condition 5.

$$M'_{n-1} \cap F_c^{\perp} = (M'_{\underline{r}+1} + V_{fm}) \cap F_c^{\perp} = (M_{\underline{r}} + L + V_{fm}) \cap F_c^{\perp}.$$

So

$$\begin{aligned} & (M'_{n-1} \cap F_c^{\perp}) \subset (F'_{c+1})^{\perp} \\ \iff & F'_{c+1} \subset ((M_{\underline{r}} + L + V_{fm}) \cap F_c^{\perp})^{\perp} \\ \iff & F'_{c+1} \subset (M_{\underline{r}} + L + V_{fm})^{\perp} + F_c \\ \iff & F'_{c+1} \subset (M_r \cap L^{\perp} \cap F_{fm}^{\perp}) + F_c. \end{aligned}$$

Note that $L \subset F_c^{\perp}$ so $F_c \subset L^{\perp}$. So we have

$$F'_{c+1} \subset (M_r \cap L^{\perp} \cap V_{fm}^{\perp}) + F_c \subset (L^{\perp} + F_c) = L^{\perp}$$

which implies $L \subset F'_{\underline{c+1}}$.

- Now define F'_j for $c + 2 \le j \le n$ as

$$F'_j = F'_{c+1} + (F'_{\underline{c+1}} \cap M'_{r_j})$$

where r_j is the row of the black checker in column j of the $\bullet$-configuration. With perps, this gives $F'_{\cdot}$.

With $\mu = 0$ we get the original point $(V_., M_., F_.)$, so $\mathcal{F} \not\subset D_Q$. With $\nu = 0$, $V'_d = V'_t$ so $(V'_., M'_., F'_.) \in D_{\text{NW good quad}}$ and $\mathcal{F}$ meets $D_{\text{NW good quad}}$. We will see that D_Q contains the divisor $\nu = 0$ with multiplicity 1, proving the result.

The divisor D_Q on $\mathcal{F}$ is given by requiring

1. $V_{\mathbf{a}} \subset (F'_{c+1})^{\perp}$
2. $V_{m(F_{c+1})} \subset F'_{c+1}$ and
3. $(M'_n \cap F_c^{\perp}) \subset F'_{\underline{c+1}}$.

Note that $(M'_{n-1} \cap F_c^{\perp}) \subset F'_{\underline{c+1}}$ along with conditions 1 and 2 are satisfied by all points of $\mathcal{F}$. A point of $\mathcal{F}$ is in D_Q

$$\begin{aligned}
&\iff (M'_n \cap F_c^{\perp}) \subset F'_{\underline{c+1}}\\
&\iff ((M_{\underline{r}+1} + V'_{gm}) \cap F_c^{\perp}) \subset F'_{\underline{c+1}}\\
&\iff ((M_{\underline{r}} + L + V_{fm} + \langle \mu e_t + \nu e_d\rangle) \cap F_c^{\perp}) \subset F'_{\underline{c+1}}\\
&\iff (M_{\underline{r}} + V_{fm} + \langle \mu e_t + \nu e_d\rangle) \cap F_c^{\perp} + L \subset F'_{\underline{c+1}}\\
&\iff [(M_{\underline{r}} + V_{fm}) + \langle \mu e_t + \nu e_d\rangle] \cap F_c^{\perp} \subset F'_{\underline{c+1}}.
\end{aligned}$$

The last equivalence is because $L \subset F'_{\underline{c+1}}$.

Let $\ell = \dim((M_{\underline{r}} + V_{fm}) \cap F_c^{\perp})$. Choose a basis $e_1, \ldots, e_k$ for $F_c^{\perp}$ and $f_1, \ldots, f_j$ for $M_{\underline{r}} + V_{fm}$ such that $e_i = f_i$ for $1 \le i \le \ell$. Then

$$(M_{\underline{r}} + V_{fm}) + F_c^{\perp} = \langle e_1, \ldots, e_k, f_{\ell+1}, \ldots, f_j\rangle.$$

Define a projection

$$\sigma : (M_{\underline{r}} + V_{fm}) + F_c^{\perp} \to F_c^{\perp}$$

by $\sigma(e_i) = e_i$ for $1 \le i \le k$ and $\sigma(f_i) = 0$ for $\ell + 1 \le i \le j$. The projection σ vanishes on $(M_{\underline{r}} + V_{fm})/[(M_{\underline{r}} + V_{fm}) \cap F_c^{\perp}]$ so $(Id - \sigma)((M_{\underline{r}} + V_{fm}) + F_c^{\perp}) \subset M_{\underline{r}} + V_{fm}$. Note that σ is fixed for all points in the family $\mathcal{F}$. So we can continue the equivalence:

$$\begin{aligned}
&[(M_{\underline{r}} + V_{fm}) + \langle \mu e_t + \nu e_d\rangle] \cap F_c^{\perp} \subset F'_{\underline{c+1}}\\
&\iff [(M_{\underline{r}} + V_{fm}) + \langle \sigma(\mu e_t + \nu e_d)\rangle] \cap F_c^{\perp} \subset F'_{\underline{c+1}}\\
&\iff (M_{\underline{r}} + V_{fm}) \cap F_c^{\perp} + \langle \sigma(\mu e_t + \nu e_d)\rangle \subset F'_{\underline{c+1}} \quad \text{since } \sigma(\mu e_t + \nu e_d) \in F_c^{\perp}\\
&\iff \sigma(\mu e_t + \nu e_d) \in F'_{\underline{c+1}} \quad \text{since } M_{\underline{r}} + V_{fm} \subset M'_{n-1} \text{ and } M'_{n-1} \cap F_c^{\perp} \subset F'_{\underline{c+1}}\ \forall \text{ pts of } \mathcal{F}\\
&\iff \mu\sigma(e_t) + \nu\sigma(e_d) \in F'_{\underline{c+1}}\\
&\iff \mu e_t + \nu\sigma(e_d) \in F'_{\underline{c+1}} \quad \text{since } \sigma \text{ is the identity on } F_c^{\perp}\\
&\iff \nu\sigma(e_d) \in F'_{\underline{c+1}} \quad \text{since } t \prec a, \text{ so } \langle e_t\rangle \subset V_{\mathbf{a}} \subset F'_{\underline{c+1}} \quad \forall \text{ pts of } \mathcal{F}.
\end{aligned}$$

The final statement is true only if we are in the divisor D_Q. Since $\mathcal{F} \not\subset D_Q$, this statement is not satisfied by all points in $\mathcal{F}$.

Thus the restriction of D_Q to $\mathcal{F}$ has two components, each with multiplicity 1. They are:

1. the hyperplane section $\{F'_{c+1} \mid \sigma(e_d) \in F'_{\underline{c+1}}\} \subset \mathcal{F}$
2. the fiber for $\nu = 0$ is also a component, appearing with multiplicity 1 as desired. □

Connecting divisors to white checker moves. We have two loose ends to tie up to conclude the proof of quadratic degenerations in the type B_n geometric Littlewood–Richardson rule. These loose ends are exactly Section 5.16 of [14]. We restate them here nearly verbatim.

The loose ends:

1. $\pi(D_\emptyset) = \overline{X}_{\circ_{stay}\bullet_{next}}$ and / or $\pi(D_{\text{NW good quad}}) = \overline{X}_{\circ_{swap}\bullet_{next}}$.
2. Furthermore $\overline{X}_{\circ_{stay}\bullet_{next}}$ appears with multiplicity 1 in $Cl_{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet}$ if $D_\emptyset$ appears with multiplicity 1 in $Cl_{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}X_{\circ\bullet}$, and similarly for $\overline{X}_{\circ_{swap}\bullet_{next}}$ and $D_{\text{NW good quad}}$.

Both are a consequence of the next result ((2) using the fact that π is birational).

Theorem 4.7. *The morphism π induces birational maps from*

(a) $D_\emptyset$ *to* $\overline{X}_{\circ_{stay}\bullet_{next}}$ *and*

(b) $D_{NW\,good\,quad}$ *to* $\overline{X}_{\circ_{swap}\bullet_{next}}$.

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